Conceptions of Infinity
An APOS Analysis

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Abstract:- The present study investigates the conceptions of infinity of students in some official secondary schools. A questionnaire was administrated to a group of 215 students of grade 10 and 11. After checking the presentation of infinity in the Lebanese curriculum, a comparative study was made between the two grades. Results showed that the majority of students conceptualize infinity as *endless* or *huge number.* Students hold misconceptions in differentiating between infinite and innumerable and in relating the infinite to unbounded. APOS results showed a strong association between process and object thinking and showed that students' APOS levels vary according to the context in which infinity is given.

***Keywords**:- Infinity, Misconception, APOS Theory, Curriculum, Actual and Potential Infinity.*

I. INTRODUCTION

The concept of infinity is one of the abstract concepts in Mathematics. This is due to the duality in its meaning: a process or an object. Even great mathematicians and philosophers through history couldn't grasp the infinite. Infinity is no doubt related to many important concepts in mathematics such as number configuration, infinite sets, limits, continuity, sequences and calculus in general. Therefore, misconceptions of infinity could lead to misconceptions in many fields of mathematics.

Previous research report that students and some pre-service teachers hold misconceptions and have ambiguities in infinity. Moreover, as a mathematics teacher for secondary levels, the researcher found that students also have discomfort in this concept although they are able to solve some tasks involving infinity. Starting from here, the researcher was curious to know how students conceptualize infinity? What do they think it is? How do they view it? Do they think it is a process? Object? Can they differentiate between innumerable and infinite? Do they relate the unbounded to the infinite? Are they convinced that an infinite of numbers is embedded in a bounded interval?

The purpose of this study is to explore students' conceptions and especially misconceptions of infinity, in the light of the Lebanese curriculum and in accordance to the APOS theory. In particular, the aim of this research is to compare students' conceptions of infinity before and after it is presented in the program to study its impact on students' conceptions.

This research raises the following questions:

- How do students understand the concept of infinity according to the APOS theory?
- What kinds of misconceptions do they form?
- How does the Lebanese curriculum present the infinity concept? What is the impact of the curriculum on the students' conception of infinity?

II. THEORETICAL FRAMEWORK

A. Historical Background

After revising the history, one can notice that the human mind couldn't accept the actual infinity that easy. Pythagoras argued that the universe was made of finite natural numbers (in O'Connor & Robertson, 2002). Aristotle argued against the actual infinity and believed in a finite universe (in O'Connor & Robertson, 2002). Similarly, Euclid didn't consider the actual infinity (in O'Connor & Robertson, 2002). In the middle of the 15th century, Galileo differentiated between "potential infinity" and "actual infinity", where actual infinity stands for viewing infinity as an object and potential infinity stands for viewing infinity as a process. He also produced the one-to-one correspondence between the positive integers and their squares (in O'Connor & Robertson, 2002). In the middle of the 16th century, the symbol of infinity was first introduced (in Whinston, 2009), and during the second half of the 17th century, a big development with infinity took place with the invention of the differential and integral calculus (in Clouse, 2002). In the 19th century, Cantor was the first to do a major work with different sizes of infinity and by this, he was the first one to accept the actual infinity (in Clouse, 2002). In the middle of the 20th century, when Robinson introduced the theory of "non-standard analysis", infinitesimals were put on rigorous mathematical basis (in Tall & Tirosh, 2001).

B. Theoretical Background

- *The APOS Theory*

The APOS theory is a constructive theory of learning based on Piaget's theory of reflective abstraction and applied to learning mathematical concepts for undergraduate level (Dubinsky & McDonald, 2001).

The acronym APOS, denoting **A**ction, **P**rocess, **O**bject, **S**chema, refers to the types of mental structures an individual build in responding to certain problem solving situations. An individual uses certain mental mechanisms, such as *interiorization*, *coordination*, and *encapsulation* to construct these structures (Dubinsky et al., 2008). According to the APOS theory, the formation of a mathematical concept begins by applying transformations on existing mental objects. The transformation which is triggered by memory or step-by-step instruction is termed as *action*. Upon repeating and reflecting on an action, this action could be interiorized to a *process* (Dubinsky & McDonald, 2001).

A process is a mental structure that performs transformation of objects but in the mind of an individual without executing each step explicitly. When an individual becomes aware of the process as a totality and realizes that transformations can be applied to it and is able to construct such transformations, this individual is said to have *encapsulated* the process into a cognitive object (Dubinsky & McDonald, 2001). The collection of *actions*, *processes*, *objects*, and other *schemas* form the *schema* of a certain mathematical concept. Schemas are considered a framework which is used in a problem situation involving that concept (Dubinsky & McDonald, 2001).

C. Concept Image and Concept Definition Theory

In this theory, Tall and Vinner (1981), pointed out that an individual has intuitive vision about various mathematical concepts which differ from the formal definitions of these objects. The term *concept image* is used "to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981). The term *concept definition* is defined as: "a form of words used to specify a certain mathematical concept" (Tall & Vinner, 1981). According to Tall & Vinner, conflicts could occur when there is no coherence between the *concept image* and the *concept definition* of a certain mathematical concept.

III. LITERATURE REVIEW

Tall (2001b) considered that the concept of infinity arises by reflecting on finite experiences and extending them to infinite. According to Tall (1981), a child first realizes that the counting process is endless. Later, when the theoretical notation is considered with the symbol ℕ being introduced, the totality of natural numbers arises.

In another study done on the historical issues concerning infinity (2005), Dubinsky and his colleagues pointed out that the human mind tends to accept the potential infinity rather than the actual one. In the same paper, Dubinsky reported that below the age of 15, there are no signs of thinking of actual infinity. Fischbein (in Dubinsky, 2005) argued in reference to actual infinite sets and noted that it is "contradictory in natural intuitive terms".

In a three-year study done on grade 5 till 8 students on the development of understanding of infinity, Pehkonen &al. (2006) found that as individuals grow, their ability to understand infinity gets better. Moreover, the authors found out that potential infinity is realized before actual infinity. Contrary to this study, Singer &Voica (2003) showed that there is no connection between age and deep understanding of infinity.

A study on teachers' perception of infinity done by Kattou (2010) revealed that the majority of teachers comprehend infinity as an endless process. Dubinsky &al. (2008) found out that students have difficulties in solving problems containing infinite iterative processes. They claim that the student's ability to make progress towards a solution depended on the ability to see the underlying iterative processes as completed totalities.

Dubinsky, Weller, McDonald, & Brown (2005) indicated possible existence of three conceptions: small finite (that which can be physically and cognitively completed), large finite (that which cannot be physically completed but cognitively completable), and the infinite (that which is physically uncompletable and cognitively completable, only when mental constructions that are not required in the finite case are made). The authors presented results of some previous research which show that students mix the concepts: large finite and infinite. They claim that the *encapsulation* is an important step to overcome these difficulties.

Tall (2001a) advises the teachers to trust their own intuitions more than the formal rigor mathematics. This way of teaching appeals to be natural and is at the same level of the student's mind. He suggests providing students with rich experiences in order to build a coherent concept. This involves a balance between the variety of examples and non-examples. On the other hand, Voica &Singer (2003), suggest early informal preparation for this concept. According to them, the approach should be made using various examples from different contexts and presenting various opinions of infinity outside mathematics. Moreover, they found out that the arguments of students were consistent when there were connections between algebraic and geometric thinking.

A study done by Pessia Tsamir (1999) focuses on students' concepts of cardinal infinity. She demonstrates that research-based knowledge about students' inconsistent responses to different representations of infinite sets "can be used to raise their awareness of contradictions in their own reasoning and to guide them toward using the one-to-one correspondence as the unique criterion for the comparison of infinite quantities” (in Tall & Tirosh, 2001).

IV. METHODOLOGY

The research was done in three phases. In the first phase, the parts concerning infinity of grade 10 public textbook, its teacher’s guide, and the Lebanese curriculum of Mathematics (1997) were analyzed according to the APOS theory. In the second phase, a questionnaire was administrated to 215 students (117 of grade 10 and 98 of grade 11) in 3 arbitrary chosen secondary official schools. The third phase consisted of one-to-one interviews conducted with selected participants. The gathered data was analyzed qualitatively.

A. *Students' Questionnaire*

- ➢ Define infinity in your own opinion.
- ➢ Classify the following as finite or infinite:

Write next to each number “F” for finite and “I” for infinite

- The number of grains of sand on earth. ………
- The number of possible fractions. ………
- The number of visible stars in the sky ……..
- The number of even numbers ……..
- ➢ How many numbers are there between 0.2 & 0.3? Justify.
- ➢ Suppose a magical machine gives you 1$ in the first day, and $\frac{1}{2}$ $ on the second day, and $\frac{1}{4}$ $ in the third day, and $\frac{1}{8}$ $ in the fourth day and so on…(Every day it gives you half of what it gave you the day before).Suppose that this magical machine never stops working.
- How much money will the machine give you in the tenth day?
- How much money will you have after all? $(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...)$.
- ➢ How many elements does set A contain? set B?

A={1,2,3,{0,1,2,3,…}}
B={1,2,3,{0,1,2,3}}.

It is noted that number five of this questionnaire was only included for grade 11 students since grade 10 students didn’t have knowledge in “sets” by the time the questionnaire was conducted.

V. RESULTS

A. *Results of Curriculum Analysis*

The curriculum and grade 10 textbook and its teacher’s guide analysis show that the symbol of infinity appears in grade 10 textbook for the first time without any explanation. Few notes are found in the teacher’s guide; these notes include only remarks of the importance of not using infinity as a real number and focusing on the fact that an interval contains infinity of real numbers. The approach of presenting this concept is left to the teacher. Moreover, there are few problems related to infinity in grade 10 textbook. Another noticed point was the absence of certain contexts of infinity like: sizes and comparison of infinite sets, difference between innumerable and infinite, and the difference between bounded and infinite.

B. *Results of Students' Questionnaire*

The results of question number 1 (defining infinity) revealed that the majority of students conceptualize infinity as unlimited or endless and more than half of the students defined infinity as a number. Furthermore, there is a statistical significant association between process and object thinking. Comparing grades 10 and 11, no statistical difference was found.

Results of question 2 (innumerable vs infinite) showed that the majority of students mix between innumerable and infinite and more than half of the students do not know there are infinite number of fractions. Also no significant difference between grade 10 & 11was found.

Results of questions 3,4, and 5:

Question number	APOS results	No answer	Below action level	Action level	Process level	Object level
3	Grade 10	-	3.4%	22.2%	18.8%	50.4%
	Grade 11	-	11%	25.3%	9.9%	52.7%
4	Grade 10	22%	-	-	40%	3.4%
	Grade 11	14.3%	-	-	57%	1.1%
5	Grade 11	23%	-	-	54%	23%

Table 1

It is noted that 40% of grade 11 were excluded since they had conflictions in sets.

C. Results of Interviews

The interviews conducted with selected students revealed that language hinders students from expressing their thoughts. It was also noticed that sometimes students memorize certain properties about infinity and use it without being convinced. This was evident when they were trying to explain their own answers. Moreover, some students were not sure of the answers they have written in the questionnaire and changed them during the interview.

VI. CONCLUSION

Results of the three phases of the research led to the following conclusions:

It is evident that infinity is hard to define. This result is no surprise since if not, infinity would have had a formal mathematical definition. Nevertheless, the majority of students relate infinity to huge numbers and to unbounded. The terms "unlimited" and "number" were the most frequent in the definitions given by students.

Accepting and memorizing certain properties about infinity without being convinced was evident in the interviews when students were not able try to explain their own answers and when some of them changed their minds about answers they have written in the questionnaire before. This means that their knowledge about infinity is not rigid. Students could be swaying between their own concept images of infinity (most of which are a result of their extrapolating of finite cases and may contain conflicts) and the formal mathematical properties (which they obey most of the times without deep understanding).

The influence of the explicit use of infinity in grade 10 on the student's concept image of infinity seemed negligible, since most of the results between the two grades were sensibly similar. Moreover, results show that students were able to reach the object level in questions which have similar context of infinity as their textbooks, more than questions in which infinity appears in a new context. Around half of the students reached the object level in an interval or set context, while very few of them (3.4% of Grade10 and 1.1% of grade 11) reached the object level in the infinite sum context. The reason behind this difference in the results of APOS in different contexts could be that students encounter sets and intervals, while there is absence of infinite sums in grade 10 program. This shows the importance of exposing the student to a variety of contexts involving infinity.

Results also show that there is misconception in differentiating between innumerable and the infinite. According to the Singer & Voica (2003), the human mind usually correlates huge numbers with infinity. Nevertheless, there is absence of this point in the curriculum. This could be one of the reasons for having this type of misconception in infinity. Another reason could be related to the axis. Students usually label some numbers from the left and right of the origin and they put an arrow that indicates that numbers are getting larger (to the right) and smaller (to the left). This could trigger the correlation between the very large finite numbers which they can't label and the infinite. Similarly, upon writing an infinite set with some of its elements and then three points like: { 0, 1, 2, 3,…}, students could think that the big numbers and infinity are the same.

Results of the research show the domination of the process concept image of students. Also, the struggle in their minds between the process and object conception of infinity is shown. The statistical work shows a significant correlation between the process and object. This shows how hard it is to conceive infinity as an object. On the other hand, it is noticed that the curriculum presents the infinity concept by providing its symbol '∞' only without being aware of the struggle that happens in the student's mind.

The analysis done throughout this research raises questions about the roots of ambiguities and difficulties in infinity and the reasons behind them. Of course, infinity is a hard abstract concept. But maybe if the concept images of students were guided from early stages by providing richer experiences and giving them the chance to explore this concept in different contexts, their concept images of infinity would contain less conflictions. Maybe encouraging mathematical reasoning rather than acceptance and of the mathematical properties, would trigger deeper thinking of abstract issues, particularly infinity. More connections between different forms and contexts of infinity may be required in order to form a network with smooth paths between these different forms in students' minds as Tall (1988) suggested.

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