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Generalized SOR Method for solving Non-Square Linear Systems

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Abstract:- In this paper, we propose SOR method for the solution of non-square linear systems which can be called as generalized SOR method. A numerical example is considered to exhibit the superiority of this method over the generalized Jacobi and generalized Gauss-Seidel methods. AMS Subject Classification:- 15A06,65F15,65F20,65F50

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I. INTRODUCTION

Without loss of generality, we consider a linear system of 'm' equations in 'n' unknowns (m<n) of the form

$$x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \qquad (1.1)$$

These equations can be expressed as a matrix system

$$AX = b \qquad ..(1.2)$$

where $A \in \mathbb{R}^{m\times n}$ and X and b are unknown and known m and n dimensional vectors respectively.
Partitioning the rectangular matrix A as

$$A = [B\ \widetilde{B}] \qquad ..(1.3)$$

where $B \in \mathbb{R}^{m\times n}$ and $\widetilde{B} \in \mathbb{R}^{m,n-m}$, then the equation (1.2) can be put in the form

$$[B\ \widetilde{B}]\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = b \qquad ..(1.4)$$

where X_1 and X_2 are m and $n-m$ dimensional vectors respectively.
Solving (1.2) is same as solving

$$BX_1 + \widetilde{B}X_2 = b \qquad ..(1.5)$$

For $m = n$ in (1.1), the system (1.2) it can be expressed as

$$(I - L - U)X = b \qquad ..(1.6)$$

where $-L$ and $-U$ are strictly lower and upper triangular parts of the coefficient matrix A respectively. The SOR method for solving (1.6) is defined by

$$X^{(n+1)} = (I - \omega L)^{-1}\{[(1-\omega)I + \omega U]X^{(n)} + \omega b\} \qquad ..(1.7)$$

$(n = 0,1,2,\dots)$
with the choice of relaxation parameter ω *as*

$$\omega = \frac{2}{1+\sqrt{1+\bar{\mu}^2}} \qquad ..(1.8)$$

where $\bar{\mu} = max|\mu_i|$ and μ_i are the eigenvalues of the Jacobi matrix $J = L + U$.

Using the iterative method developed by Wheaton and Samuel[1], Saha[2] proposed the generalized Jacobi method for solving non-square linear systems, and he also proposed generalized Guass-Seidal method which looks somewhat hollow as the inverse of a strictly lower triangular matrix doesn't exist.

In this paper, we describe the generalized SOR method in section 2 and in section 3 its convergence criteria is discussed. A numerical example is presented in the concluding section to show the superiority of the proposed method.

II. GENERALIZED SUCCESSIVE OVER RELAXATION (GSOR)

➢ *Method*

Let $X^{(k)}$ be the k^{th} approximate solution to the system(1.2). Then as mentioned in [1] , the (k+1) approximate is obtained as

$$X^{(k+1)} = X^{(k)} + S(A)d^{(k)} \qquad ..(2.1)$$

where S(A) is the $n \times m$ matrix having $0, -1$ and 1 as its elements indicating the signs of the transpose matrix of A and

$$d^{(k)} = \left[\frac{b_1 - A_1X^{(k)}}{m\|A_1\|}, \frac{b_2 - A_2X^{(k)}}{m\|A_2\|}, \dots\dots \frac{b_m - A_mX^{(k)}}{m\|A_m\|}\right]^T$$
..(2.2)

Here $\|A_i\|, (i = 1,2,,,,m)$ are l_1-norms of i^{th} row vector of A.
Do the following steps by considering the equation (1.4) with an initial approximation $X^{(0)} = \begin{pmatrix} X_1^{(0)} \\ X_2^{(0)} \end{pmatrix}$.

Step 1: Assign $k \leftarrow 0$

Step 2: Calculate for a positive definite matrix B,

$$\tilde{b}^{(k)} = b - BX_1^{(k)} \qquad ..(2.3)$$

Applying the procedure given in [1] to the system $\widetilde{B}Y = \tilde{b}^{(k)}$ with the initial guess $X_2{}^{(k)}$ to obtain $X_2{}^{(k+1)}$ *i.e.,*

$$X_2{}^{(k+1)} = X_2{}^{(k)} + S(\widetilde{B})\, d^{(k)} \qquad ..(2.4)$$

where $S(\widetilde{B})$ is a matrix of order $(n-m)\times m$ whose elements are the signs of the elements of $\widetilde{B}^T$ and $d^{(k)}$ is m-dimensional vector whose i^{th} entry is

$$d_i{}^{(k)} = \frac{\tilde{b}_i{}^{(k)} - \widetilde{B}_i X_2{}^{(k)}}{m\|\widetilde{B}_i\|} \qquad ..(2.5)$$

Step 3: Calculate

$$\hat{b}^{(k)} = b - \widetilde{B}X_2{}^{(k+1)} \qquad ..(2.6)$$

Applying the SOR method (1.7) with (1.8) to the square linear system $Bz = \hat{b}^{(k)}$ with the initial guess $X_1{}^{(k)}$ to obtain $X_1{}^{(k+1)}$ i. e;

$$X_1{}^{(k+1)} = (I-\omega L)^{-1}\left[\{(1-\omega L)I + \omega U\}X_1{}^{(k)} + \omega\hat{b}^{(k)}\right] \qquad .. (2.7)$$

$$(\text{or})\ X_1{}^{(k+1)} = L_\omega X_1{}^{(k)} + (I-\omega L)^{-1}.\,\omega\hat{b}^{(k)} \qquad ..(2.8)$$

where $L_\omega = (I-\omega L)^{-1}\{(1-\omega)I + \omega U\}$ is the SOR iteration matrix and 'ω' is as given in (1.8).

Step 4: Obtain $X^{(k+1)} = \begin{pmatrix} X_1{}^{(k+1)} \\ X_2{}^{(k+1)} \end{pmatrix}$ where $X_2{}^{(k+1)}$ and $X_1{}^{(k+1)}$ are obtained in steps 2 and 3 respectively.

Step 5: If $\|AX^{(k+1)} - b\| < \epsilon$ where ϵ is the fixed threshold, take $X^{(k+1)}$ as the solution of (1.2). If not assign $k \leftarrow k+1$ and go to step 2.

Remark

For $\omega = 1$ in (2.7), the generalized SOR method realizes generalized Gauss-seidal method and for the generalized Jacobi method, replace the equation (2.7) by

$$X_1{}^{(k+1)} = (L+U)X_1{}^{(k)} + \hat{b}^{(k)} \qquad ..(2.9)$$

III. CONVERGENCE CRITERIA

Let $X^{(k)} = \begin{pmatrix} X_1{}^{(k)} \\ X_2{}^{(k)} \end{pmatrix}$ where $X_1{}^{(k)} \in \mathbb{R}^m$ *and* $X_2{}^{(k)} \in \mathbb{R}^{n-m}$, be the k^{th} appropriate solution of the system (1.4)

As

$$d_i{}^{(k)} = \frac{\tilde{b}_i{}^{(k)} - \widetilde{B}_i X_2{}^{(k)}}{m.\|\widetilde{B}_i\|}$$

$$= \frac{b_i - B_i X_1{}^{(k)} - \widetilde{B}_i X_2{}^{(k)}}{m.\|\widetilde{B}_i\|}$$

$$= \frac{b_i - (B_i X_1{}^{(k)} + \widetilde{B}_i X_2{}^{(k)})}{m.\|\widetilde{B}_i\|}$$

$$= \frac{b_i - A_i X_1{}^{(k)}}{m.\|\widetilde{B}_i\|} \qquad ..(3.1)$$

If we denote the non-singular matrix $\text{diag}(\|\widetilde{B}_1\|, \|\widetilde{B}_2\|, \ldots.. \|\widetilde{B}_m\|)$ by $N(\widetilde{B})$, then (3.1) can be put in the form

$$d^{(k)} = \frac{1}{m} N(\widetilde{B})^{-1}(b - AX^{(k)}) \qquad ..(3.2)$$

By the equation (2.4) of step2 of the algorithm, we have

$$X_2{}^{(k+1)} = X_2{}^{(k)} + \frac{1}{m} S(\widetilde{B}).N(\widetilde{B})^{-1}(b - AX^{(k)}) \qquad ..(3.3)$$

With this the equation (2.6) of step3 further reduces to

$$\hat{b}^{(k)} = b - \widetilde{B}X_2{}^{(k)} + \frac{1}{m} S(\widetilde{B}).N(\widetilde{B})^{-1}(AX^{(k)} - b) \qquad ..(3.4)$$

The SOR method now for obtaining $X_1{}^{(k+1)}$ is derived from (2.7) can be rewritten as

$$\begin{aligned} X_1{}^{(k+1)} &= (I-\omega L)^{-1}\left[(I - \omega L + \omega I + \omega L + \omega U)X_1{}^{(k)} + \omega\hat{b}^{(k)}\right] \\ &= (I-\omega L)^{-1}\left[(I-\omega L)X_1{}^{(k)} - \omega\{(I - L - U)X_1{}^{(k)} - \hat{b}^{(k)}\}\right] \\ &= X_1{}^{(k)} - \omega(I-\omega L)^{-1}\{BX_1{}^{(k)} - \hat{b}^{(k)}\} \qquad ..(3.5) \end{aligned}$$

Since $AX^{(k+1)} = BX_1{}^{(k+1)} + \widetilde{B}X_2{}^{(k+1)}$, from (3.3), (3.4) and (3.5) we can have

$$\begin{aligned} AX^{(k+1)} &= BX_1{}^{(k+1)} + \widetilde{B}X_2{}^{(k+1)} \\ &= BX_1{}^{(k)} - \omega B(I-\omega L)^{-1}\Big\{BX_1{}^{(k)} - b + \widetilde{B}X_2{}^{(k)} - \frac{1}{m} S(\widetilde{B}).\widetilde{B}.N(\widetilde{B})^{-1}(AX^{(k)} - b)\Big\} \\ &\quad + \widetilde{B}X_2{}^{(k)} + \frac{1}{m} S(\widetilde{B}).\widetilde{B}.N(\widetilde{B})^{-1}(b - AX^{(k)}) \\ &= AX^{(k)} - \omega B(I-\omega L)^{-1}\{(AX^{(k)} - b) - P(AX^{(k)} - b)\} - P(AX^{(k)} - b) \end{aligned}$$

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where

$$P = \frac{1}{m}.\tilde{B}.S(\tilde{B})N(\tilde{B})^{-1} \qquad ..(3.6)$$

Now,

$$AX^{(k+1)} - b = (AX^{(k)} - b) - \omega B(I - \omega L)^{-1}(I - P)(AX^{(k)} - b) - P(AX^{(k)} - b)$$

$$= (AX^{(k)} - b)[I - \omega B(I - \omega L)^{-1}(I - P) - P]$$

$$= (AX^{(k)} - b)[\{I - \omega B(I - \omega L)^{-1}\}.(I - P)] \qquad ..(3.7)$$

From (3.8), we have $X = \lim_{k} X^{(k)}$ is a solution of the system (1.4) provided

$$\|I - \omega B(I - \omega L)^{-1}\| < 1 \qquad ..(3.8)$$

and

$$\|(I - P)\| < 1 \qquad ..(3.9)$$

where P is the matrix as given in (3.6).For the existence of x, we consider

$$\|X^{(k+1)} - X^{(k)}\| = \|X_1^{(k+1)} - X_1^{(k)}\| + \|X_2^{(k+1)} - X_2^{(k)}\|$$

$$.\|\omega(I - \omega L)^{-1}(I - P)(b - AX^{(k)})\| + \left\|\frac{1}{m} S(\tilde{B})N(\tilde{B})^{-1}(b - AX^{(k)})\right\|$$

$$\leq \|\omega(I - \omega L)^{-1}(I - P)\| + \frac{1}{m}\left\| S(\tilde{B})N(\tilde{B})^{-1}\right\| \|AX^{(k)} - b\| \qquad ..(3.10)$$

With this we can conclude that $X = \lim_{k} X^{(k)}$ exists as $X^{(k)}$ is a Cauchy-sequence.

Hence, the generalized SOR method converges under the conditions (3.8) and (3.9) for any matrix norm.

Remark

It can be shown that the Generalized Guass-seidal and Generalized Jacobi methods converge under the conditions.

$$\|I - B(I - L)^{-1}\| < 1 \text{ and } \|I - B\| < 1 \qquad ..(3.11)$$

respectively along with the condition (3.9), as done above.

IV. NUMERICAL EXAMPLE

In this section, we consider the following non-square linear system i.e;

$$\begin{bmatrix} 1 & -2/5 & 0 & -1/5 & 3/5 & 1/5 & 2/5 \\ -14/35 & 1 & -2/7 & 0 & 4/10 & 4/10 & -3/10 \\ 0 & -14/35 & 1 & -1/5 & 3/5 & -3/5 & 1/5 \\ -1/5 & 0 & -1/5 & 1 & -2/5 & 3/5 & 1/5 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} 530/105 \\ 11/210 \\ 134/35 \\ -289/105 \end{pmatrix}$$

This system can be expressed as in (1.5), as

$$\begin{bmatrix} 1 & -2/5 & 0 & -1/5 \\ -14/35 & 1 & -2/5 & 0 \\ 0 & -14/35 & 1 & -1/5 \\ -1/5 & 0 & -1/5 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ p \end{pmatrix} + \begin{bmatrix} 3/5 & 1/5 & 2/5 \\ 4/10 & 4/10 & -3/10 \\ 3/5 & -3/5 & 1/5 \\ -2/5 & 3/5 & 1/5 \end{bmatrix} \begin{pmatrix} q \\ r \\ s \end{pmatrix} = \begin{pmatrix} 530/105 \\ 11/210 \\ 134/35 \\ -289/105 \end{pmatrix}$$

It is to note that the conditions for convergence criteria i.e; (3.8), (3.9) and (3.11) are satisfied for the above system. Now applying the procedure given in section 2, we obtained the following data and the non-basic solutions upto an accuracy of 0.5×10^{-10} starting with a null vector as a initial guess.

n	Values of $\|b - AX\|_1$ in successive iterations and obtained final solution upto an error less than 0.5E-10		
	Generalised JACOBI	Generalised GAUSS-SEIDAL	Generalised SOR
1	4.09352609	1.83483632	1.08856249
2	1.87329034	0.51129288	0.29789633
3	0.72835033	0.16122001	0.08271513
. .	. .	. .	. .
16	10.3774186E -6	4.25251637E-8	10.3810175E-9
17	3.9486858E- 6	1.93810483E-8	10.3810175E-9
. . .	. . .	. . .	(x,y,z,p,q,r,s) =(5.01262429, 2.96595439, 2.06287791 0.87928128, 2.42931515, -2.42931515, 1.06376678)
22		7.7626535E-9	
23		7.7626535E-9	
		(x,y,z,p,q,r,s)=(4.99214357,2.93471021, 1.99688395, 0.91817848, 2.48270963, -2.48270963, 1.04977851)	
.	.		
.	.		
30	11.09537E-9		
31	11.09537E-9		
	(x,y,z,p,q,r,s) = (3.19440836, 1.12036248, 2.59066879, - 1.05611782, 2.62982539, 0.51312802, 1.02402821)		

Table 1

V. CONCLUSION

It is evident from the above table-1 that the GSOR method converged more rapidly than the Generalised Jacobi and Generalised Gauss-seidal methods for solving non-square linear systems as in the case of SOR method with respect to Jacobi and Gauss-seidal methods for the solution of square linear systems.

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