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Global Dynamics of a PDE Model for Eradication of Invasive Species

Rana D. Parshad
Department of Mathematics,
Iowa State University,
Ames, IA 50011, USA.

Said Kouachi
Department of Mathematics,
College of Science, Qassim University,
P.O.Box 6666, Buraydah 51452, KSA.

Jingjing Lyu
Department of Mathematics,
Clarkson University
Potsdam NY13699, USA.

Abstract:- The purpose of the current manuscript is to propose a generic method that causes the local extinction of a harmful invasive species. Eradication is achieved via introduction of phenotypically modified organisms into a target population. Here we propose a model without the logistic type term, of which the reaction terms may change sign, and so the solutions are not bounded a priori. We prove global existence of solutions via a Lyapunov function method, and show existence of a finite dimensional (L2(Ω);H2(Ω)) global attractor that supports states of extinction, improving current results in the literature. We also conduct numerical simulations to investigate the decay rate of the female species. Lastly we apply optimal control techniques to compare the effectiveness of various reaction terms on species extinction.

***Keywords*:-** *Reaction Diffusion System, Global Existence, Global Attractor, Optimal Control, Invasive Species, Biological Control.*

I. INTRODUCTION

An exotic species commonly referred to as invasive species, is any species capable of propagating into a non-native environment. As a result of globalization, exotic species are being introduced to ecosystems around the world at an unprecedented pace, in many cases causing harm to the environment, human health, and/or the economy [35, 34]. Once an exotic species is established in a new environment, its detrimental potential might be realized in the form of economic losses or threats to public health. Eradication initiatives in these cases frequently require continuous efforts for long periods of time. A small fraction of the estimated 50,000 exotic species in the US is harmful, but they inflict considerable damage [50, 16]. Studies indicate losses of about $120 billion/year by 2004 [50]. A strategy for eradication of exotic species in which a \Trojan individual" is strategy is relevant to species with an XY sex-determination system, in which males are the heterogametic sex (carrying one X chromosome and one Y chromosome, XY) and females are the homogametic sex (carrying two X chromosomes, XX).

Variations in the sex chromosome number can be produced through genetic manipulation; for example, a phenotypically normal and fertile male fish bearing two Y chromosomes termed supermales (s) [2, 6, 7, 8]. Additional manipulations through hormone treatments can reverse the sex, resulting in a feminized YY supermale [36, 33, 22]. The eradication strategy involves the addition of sex-reversed females bearing two Y chromosomes, i.e. feminized supermales (r), at a constant rate μ to a target population containing f and m. Mating between the introduced r and the wild-type m generates a disproportionate number of males over time. The higher incidence of males decrease the female to male ratio. Ultimately, the number of f decline to zero, causing local extinction. This theoretical method of eradication is known as Trojan Y Chromosome strategy (TYC), [13]. Note, if an invasive species is used as a biological weapon, one would aim at maximum damage, by choosing a species that might populate very rapidly, and not grow according to the logistic control terms (at least in certain time windows), assumed traditionally [51, 52]. There is a large literature of such rapid population actuations in the so called case of an insect \outbreak" [4]. Furthermore, past models have not considered the effects of directed movements, such as movement of the males and supermales towards high concentrations of females, or avoidance of high concentrations, of each other. Thus such situations also need to be considered in our setting.

The TYC model has been intensely investigated recently [13, 42, 41, 43, 14, 66, 63, 59, 64, 44, 45, 61, 15], and in the case of the classical TYC model, we now know the attractor is actually in Hs, 8s _ 0, [66]. However, a number of fundamental questions remain unanswered concerning existence of solutions as well as the existence and regularity of a global attractor, in the case that the reaction terms are \bad", that is say without logistic control terms, so that no a priori bounds on the solutions are possible. In [46] we began a program where we study TYC type models for biological control, where we remove the logistic type term. We also assume nonlinear and functionally dependent birth and death rates, instead of the constant coefficient birth and death rates, assumed earlier. In this case the system poses serious mathematical difficulties, as the nonlinearities change sign, and the components of the solution are not priori bounded in some Lp space. There is extensive literature on such problems [1, 18, 20, 21, 25, 39, and 65]. In [46] we were able to use an elegant Lyapunov functional to prove global existence of solutions as well prove the existence of a finite dimensional (L2(Ω);L2(Ω)) global attractor to a TYC type model. An immediate mathematical question is: Is it is possible to improve the regularity of the attractor for such a class of models? Also, from a more practical perspective one might ask, what is the decay rate of the female species?

In the current manuscript,

- We consider a major modification to the model in [46] by considering the introduction of both supermales and feminized supermales. Note in [46] we were only able to show that the global attractor of the considered system was an $(L_2(\Omega);L_2(\Omega))$. In this manuscript, we show that the attractor is actually a $(L_2(\Omega);H_2(\Omega))$ attractor, thus improving the results in [46].
- We show that extinction is always possible, under certain parameter restrictions, via the proposed strategy, even in a population which is not governed by a logistic type control term. The attractor is seen to be a one point attractor.
- We perform numerical simulations to investigate the decay rate of the female species, showing numerical evidence of exponential attraction. We also explore optimal control scenarios for extinction of the invasive species, for different reaction type terms.

II. THE MATHEMATICAL MODEL

The control method described above is modelled via the following system of reaction diffusion equations:

$$\partial_t f - d_1 \Delta f = r_1(f,m,s,r), \quad (1)$$
$$\partial_t m - d_2 \Delta m = r_2(f,m,s,r), \quad (2)$$
$$\partial_t s - d_3 \Delta s = r_3(f,m,s,r), \quad (3)$$
$$\partial_t r - d_4 \Delta r = r_4(f,m,s,r), \quad (4)$$

in $R+\times\Omega$ with the boundary conditions

$$f=m=s=r=0 \quad \text{on } R+\times\Omega \quad (5)$$

where Ω is an open bounded domain in R^n, n=1,2,3 with smooth boundary $\partial\Omega$. The functions f, m, r and s are the population densities of the normal females, normal males, supermales and sex reversed supermales respectively. The constants d_1, d_2, d_3 and d_4 are positive, called diffusion coefficients. The functions g_i, i=1,...,10 and μ are polynomials with positive coefficients. The initial data.

$$f(0,x)=m(0,x)=s(0,x)=r(0,x)=0, \quad \text{in } \Omega, \quad (6)$$

are assumed to be nonnegative and uniformly bounded on Ω. The reaction terms are given by:

$$\left\{\begin{array}{l} r_1(f,m,s,r)=\frac{1}{2}g_1(f,m,s,r)fm-g_2(f,m,s,r)f, \\ r_2(f,m,s,r)=\frac{1}{2}g_3(f,m,s,r)fm+g_4(f,m,s,r)fs \\ +\frac{1}{2}g_5(f,m,s,r)mr-g_6(f,m,s,r)r, \\ r_3(f,m,s,r)=\frac{1}{2}g_7(f,m,s,r)mr+g_8(f,m,s,r)rs-g_9(f,m,s,r)s, \\ r4(f,m,s,r)=\mu(r)-g_{10}(r)r \end{array}\right\} \quad (7)$$

Here $g_1, g_3, g_4, g_5, g_7, g_8$ are the mating rates, and g_2, g_6, g_9, g_{10} are the death rates, of the species. The function Ω is the rate of introduction of the sex reversed supermale. These coefficients are all allowed to be functionally dependent.

Equation (4) is independent of the three first equations. It is the heat equation under homogenous Dirichlet boundary conditions. Under standard conditions on the reaction term r_4:

$$\frac{\partial^2[\mu(r)-g_1(r)r]}{\partial r^2} \le 0 \text{ and } \int_{r_0^+}^{\infty} \frac{dr}{\mu(r)-g_1(r)r} = +\infty, \quad (8)$$
$$\text{where} \quad r_0^+ = \max_{x\in\bar{\Omega}} r_0(x).$$

see []. The solution of (4) with the given boundary conditions exists globally in time and is bounded on

$$\|r(t,.)\|_\infty \le r_\infty(t), \quad \text{in} \quad R^+, \quad (9)$$

where $r_\infty(t)$ is a bounded function on bounded subsets of R^+. The primary difficulty to prove the global existence of a solution to (1)-(4), is that the reaction terms given by (7) can change sign, and thus the solutions to (1)-(3) are not bounded a priori.

III. NOTATIONS AND PRELIMINARY OBSERVATIONS

For the definition of a strong solution we give the following (see for example [29])

Definition. 3.1. *We say that* $u(t,.)$: $[0,T \to L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega)$ *, is a strong solution of the system (1)-(4) if:*

i) *u is continuous on* $[0,T[$ *and* $u(0,.)=u_0(.)$.

ii) *u is absolutely continuous on compact subsets of* $]0,T[$.

iii) *u is differentiable on* $]0,T[$.

We say $u(t,.)$ *is classical if it satisfies (1)-(4) pointwise, in the usual sense of derivatives. That is we require,* .

Our aim is to construct polynomial Lyapunov functionals (see S. Kouachi and A. Youkana [25] and S. Kouachi [26, 27]) involving the solutions (f,m,s) of system (1)-(3), so that we may estimate their L^p–bounds and deduce global existence.

The usual norms in spaces $L^p(\Omega)$, $L^\infty(\Omega)$ and $C(\bar{\Omega})$ are respectively denoted by

$$\|u\|_p^p = \frac{1}{|\Omega|}\int_\Omega |u(x)|^p dx, \quad (10)$$

and

$$\|u\|_\infty = \max_{x\in\bar{\Omega}} |u(x)|. \qquad (11)$$

Since the nonlinear right hand side of (1)-(4) is continuously differentiable on R_+^4, then for any initial data in $C(\overline{\Omega})$ or $L^p(\Omega), p\in (1,+\infty)$, it is easy to check directly its Lipschitz continuity on bounded subsets of the domain of a fractional power of the operator

$$\begin{pmatrix} -d_1\Delta & 0 & 0 & 0 \\ 0 & -d_2\Delta & 0 & 0 \\ 0 & 0 & -d_3\Delta & 0 \\ 0 & 0 & 0 & -d_4\Delta \end{pmatrix}. \qquad (12)$$

It is well known that to prove global existence of solutions to (1)-(3) (see, for example [19]), there are several methods such as the method of comparison with corresponding ordinary differential equations, method of invariant regions and functional methods based on a priori estimates. This last method, implies in several cases the global existence in time by application (to the reaction terms) of the well known regularizing effect (see for example [9]) which is also called L^p–L^∞ smoothing effect of the heat operator (**i.e. the diffusion equation has an instantaneous regularizing effect in the sense that the above solution** u **belongs to** $L^\infty\left([0,T_{\max}[\,L^\infty(\Omega)\right)$ **regardless of the regularity of the initial data and that of the reaction to belong to** $L^\infty\left([0,T_{\max}[\,L^p(\Omega)\right)$ **for some** $p>N/2$). The proof is based on the Riesz-Thorin interpolation Theorem (see e.g. [11]). Rigorously it suffices to derive a uniform estimate of each $\|r_i(f,m,s,r\|_p$, , $1\le i\le 4$ on $[0,T_{\max}[$ for some $p>N/2$ and deduce that the solution to (1)-(3) is in $L^\infty(\Omega)$ for all $t\in[0,T_{\max}[$, where $T_{\max}$ denotes the eventual blow-up time in $L^\infty(\Omega)$. Under these assumptions, the following local existence result is well known (see [19,12,51,and 58]).

Proposition 1 *The system (1)-(4) admits a unique, classical solution (f,m,s,r) on* $[0,T_{\max}[\times\Omega$ *. Furthermore if* $T_{\max}<\infty$ *then*

$$\lim_{t\to T_{\max}}\left\{\|f(t,.)\|_\infty+\|m(t,.)\|_\infty+\|s(t,.)\|_\infty+\|r(t,.)\|_\infty\right\} = \infty \quad (13)$$

where $T_{\max}$ *denotes the eventual blow-up time in* $L^\infty(\Omega)$*.*

Remark 1 *In our setting a classical solution to (1)-(4) can be proved to be a strong solution. However, we refrain from this at present time.*

Remark 2 *The* ***uniqueness*** *of the solution which is a fixed point of a nonlinear operator, is obtained by using standard arguments (Fixed Point Theorem) and the fact that the reaction terms are locally Lipschitz (see for example [9]).*

IV. GLOBAL EXISTENCE

For the global existence of the system (1)-(3), we introduce the following functional used in S. Kouachi []

$$L_p(t)=\int_\Omega H_p(f(t,x),m(t,x),s(t,x))dx, \qquad (14)$$

where

$$H_p(f,m,s)=\sum_{q=0}^{p}\sum_{i=0}^{q} C_p^q C_q^i \theta_i\sigma_q f^i m^{q-i} s^{p-q}. \qquad (15)$$

The sequences $\{\theta_i\}_{i\in N}$ and $\{\sigma_q\}_{q\in N}$ are real and positive satisfying

$$\frac{\theta_i\theta_{i+2}}{\theta_{i+1}^2}\ge \bar{d}_3^{\,2}, \qquad i=1,\dots,q, \qquad (16)$$

and

$$\left(\frac{\sigma_q\sigma_{q+2}}{\sigma_{q+1}^2}-\bar{d}_1^{\,2}\right)\left(\frac{\theta_i\theta_{i+2}}{\theta_{i+1}^2}-\bar{d}_3^{\,2}\right)\ge\left(\bar{d}_2^{\,2}-\bar{d}_1^{\,2}\bar{d}_3^{\,2}\right), \qquad (17)$$

$$i=1,\dots,q, \qquad q=1,\dots,p,$$

where

$$\bar{d}_k=\frac{d_i+d_j}{2\sqrt{d_id_j}}, \qquad i\ne j\ne k, \qquad i,j,k=1,2,3.$$

(18)

Remark 3 *Conditions (16)-(17) imply that the sequences* $\left\{\frac{\theta_{i+1}}{\theta_i}\right\}_{i\in N}$ *and* $\left\{\frac{\sigma_{q+1}}{\sigma_q}\right\}_{q\in N}$ *are increasing and the sequences* $\{\theta_i\}_{i\in N}$ *and* $\{\sigma_q\}_{q\in N}$ *and can be chosen as follows*

$$\theta_i=K_\theta\left(\bar{d}_{k3}\right)^{i^2} \quad \textit{and} \quad \sigma_i=K_\sigma\left(\bar{d}_1\right)^{q^2}, \qquad (19)$$

$$i,q=0,1,\dots,$$

where K_θ *and* K_σ *are any positive constants.*

We suppose that the polynomials g_2, g_6, g_9 and g_{10} (not all constant) are sufficiently large, that is in term of limits

$$\lim_{|f|+|m|+|s|+|r|}\frac{\frac{1}{2}g_3fm+g_4fs+\frac{1}{2}g_5mr}{g_2f}<+\infty,$$

and (20)

$$\lim_{|f|+|m|+|s|+|r|}\frac{2g_6}{g_1f}=+\infty,$$

and

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$$\lim_{|f|+|m|+|s|+|r|} \frac{\frac{1}{2} g_7 mr + g_8 rs - g_9 s}{\frac{1}{2} g_6 r + \frac{1}{2} g_3 fm + g_4 fs + \frac{1}{2} g_5 mr} < +\infty, \quad (21)$$

or

$$\lim_{|f|+|m|+|s|+|r|} \frac{\frac{1}{2} g_7 mr + g_8 rs}{g_2 f + g_6 m} < +\infty, \quad (22)$$

and

$$\lim_{|f|+|m|+|s|+|r|} \frac{g_8 s}{\frac{1}{2}\frac{\theta_{i+1}}{\theta_i} g_1 fm + \frac{1}{2} g_3 fm + g_4 fs + \frac{1}{2} g_5 mr} < +\infty, \quad (23)$$

Remark 4 *Conditions (20), (21) and (22) imply that the intervals in which we choose the sequences* $\left\{\frac{\theta_{i+1}}{\theta_i}\right\}_{i\in N}$ *and* $\left\{\frac{\sigma_{q+1}}{\sigma_q}\right\}_{q\in N}$ *become sufficiently large and this gives us more* ***freedom*** *to choose the sequences.*

Remark 5 *Also note, the* $g_i(f,m,s,r)$ *, i=2,6,9,10,* ***cannot*** *all be chosen as constant. This will violate (20), (21) and (22). Note, if the* $g_i(f,m,s,r)$ *are all chosen to be constant, then for certain data* $(f_0,m_0,s_0,r_0)\in L^{\infty}(\Omega)$ *(possibly large) the solutions to problem (1)-(4) can blow-up in finite time. We demonstrate this via numerical simulation. See [30] for theoretical results on blow-up for similar systems. Also see [31] for a blow-up approach to controlling invasive populations.*

Thus we can state the following result

Theorem 4.1 *Let* $(f(t,.), m(t,.), s(t,.), r(t,.))$ *be any positive solution of the problem (1)-(4) and suppose that the polynomials* g_2, g_6, g_9 *and* g_{10} *are sufficiently large (conditions (20)-(23)), then under conditions (16)-(18) the functional* $L_p(t)$ *given by (14) is decreasing on the interval* $[0,T_{\max}[$ *.*

Proof.

Following the same reasoning as in S. Kouachi [28], that is by differentiating L_p with respect to t we got

$$L_p'(t)=\int_{\Omega}\left(\partial_f H_p \frac{\partial f}{\partial t}+\partial_m H_p \frac{\partial m}{\partial t}+\partial_s H_p \frac{\partial s}{\partial t}\right)dx$$

$$=\int_{\Omega}\left(a\partial_f H_p \Delta f+b\partial_m H_p \Delta m+c\partial_s H_p \Delta s\right)dx \quad (24)$$

$$+\int_{\Omega}\left(r_1\partial_f H_p+r_2\partial_m H_p+r_3\partial_s H_p\right)dx$$

$$=I+J.$$

Using Green's formula and the boundary conditions via(5), we obtain

$$I=-p(p-1)\int_{\Omega}\sum_{q=0}^{p-2}\sum_{i=0}^{q}\left[C_{p-2}^{q}C_q^i\left(B_{iq}T\right).T\right]f^i m^{q-i}s^{p-2-q}dx, \quad (25)$$

where the three order matrices B_{iq} are given by

$$\mathrm{B}_{\mathrm{iq}}=\begin{pmatrix} a\sigma_{q+2}\theta_{i+2} & \left(\frac{a+b}{2}\right)\sigma_{q+2}\theta_{i+1} & \left(\frac{a+c}{2}\right)\sigma_{q+1}\theta_{i+1} \\ \left(\frac{a+b}{2}\right)\sigma_{q+2}\theta_{i+1} & b\sigma_{q+2}\theta_i & \left(\frac{b+c}{2}\right)\sigma_{q+1}\theta_i \\ \left(\frac{a+c}{2}\right)\sigma_{q+1}\theta_{i+1} & \left(\frac{b+c}{2}\right)\sigma_{q+1}\theta_i & c\sigma_q\theta_i \end{pmatrix} \quad (26)$$

$$0\le i\le q, \qquad 0\le q\le p-2,$$

and T denotes the transpose vector

$$T=\left(\nabla f,\nabla m,\nabla s\right)^t.$$

From Sylvester's criterion, each of the quadratic forms (with respect to ∇f, ∇m and ∇s) associated with the matrices B_{iq}, $0\le q\le p-2$, $0\le i\le q$ is positive, if we prove the positivity of its main determinants Δ_{iq}^j, $j=1,2,3$. For a fixed $0\le i\le q$ and $0\le q\le p-2$, we see that

$$\Delta_{iq}^1=d_1\sigma_{q+2}\theta_{i+2}>0,$$

and condition (16) implies

$$\Delta_{iq}^2=d_1d_2\sigma_{q+2}^2\theta_{i+1}^2\left(\frac{\theta_i\theta_{i+2}}{\theta_{i+1}^2}-\overline{d_3}^2\right)>0.$$

We can show by elementary calculation that

$$\Delta_{iq}^2=d_1d_2d_3\sigma_{q+1}^2\theta_{i+1}^2\theta_i.\left[\overline{\theta_i}.\overline{\sigma_q}-\left(\overline{d}_2^2-\overline{d}_1^2\overline{d}_3^2\right)\right]$$

$$i=1,\ldots,q, \qquad q=1,\ldots,p,$$

where

$$\overline{\theta_i}=\left(\frac{\theta_i\theta_{i+2}}{\theta_{i+1}^2}-\overline{d}_3^2\right), \qquad \overline{\sigma_q}=\left(\frac{\sigma_q\sigma_{q+2}}{\sigma_{q+1}^2}-\overline{d}_1^2\right)$$

and this gives from (17) $\Delta_{iq}^3>0$. Consequently we have $I\le 0$.

For the second integral we have many ways to prove that $J\le 0$, but we choose only two ways: The first, since

$$I=-p(p-1)\int_{\Omega}\sum_{q=0}^{p-2}\sum_{i=0}^{q}C_q^iC_{p-2}^q\gamma_{iq}f^i m^{q-i}s^{p-1-q}dx,$$

where

$$\gamma_{iq}=\left[\frac{\sigma_{q+1}}{\sigma_q}\left(\frac{\theta_{i+1}}{\theta_i}r_1+r_2\right)+r_3\right]\sigma_q\theta_i$$

with $i=1,\ldots,q, \qquad q=1,\ldots,p$.

Replacing the reactions r_1, r_2 and r_3 by their respective values given by (7), we get

$$\frac{\gamma_{iq}}{\sigma_q\theta_i}=\frac{\sigma_{q+1}}{\sigma_q}G_i+\frac{1}{2}g_7mr+g_8rs-g_9s,$$

where

$$G_i=\frac{\theta_{i+1}}{\theta_i}\left(\frac{1}{2}g_1fm-g_2f\right)+\frac{1}{2}g_3fm+g_4fs+\frac{1}{2}g_5mr-g_6m \quad (27)$$

Then $J\leq 0$, if we choose first

$$G_i < 0,$$

which can be satisfied if we choose

$$\frac{\theta_{i+1}}{2\theta_i} g_1 fm - g_6 m < 0 < \frac{\theta_{i+1}}{\theta_i} g_2 f - g_3 12fm - g_4 fs - g_5 12mr,$$

and also this is satisfied if

$$\frac{12g_3 fm + g_4 fs + 12g_5 mr}{g_2 f} \leq \frac{\theta_{i+1}}{\theta_i} \leq \frac{2g_6 m}{g_1 fm},$$

then under condition (20), we can choose the sequence $\{\theta_i\}_{i\in N}$ satisfying (16). Secondly by choosing the sequence $\{\sigma_q\}_{q\in N}$ satisfying

$$-\frac{\sigma_{q+1}}{\sigma_q}\left(\tfrac{1}{2} g_6 m + \tfrac{1}{2} g_3 fs + \tfrac{1}{2} g_5 mr + g_4 f\right) + \tfrac{1}{2} g_7 mr + g_8 rs - g_9 s < 0,$$

which can be chosen under condition (21). The second way is that we choose

$$\frac{\sigma_{q+1}}{\sigma_q}\left(\frac{\theta_{i+1}}{\theta_i} g_1 12fm + g_3 12fm + g_4 fs + g_5 12mr\right) - g_9 s < 0 < \frac{\sigma_{q+1}}{\sigma_q}\left(\frac{\theta_{i+1}}{\theta_i} g_2 f + g_6 m\right) - g_7 12mr - g_8 rs,$$

that is

$$\frac{\frac{1}{2} g_7 mr + g_8 rs}{\frac{\theta_{i+1}}{\theta_i} g_2 f + g_6 m} \leq \frac{\sigma_{q+1}}{\sigma_q} \leq \frac{g_9 s}{\frac{1}{2}\frac{\theta_{i+1}}{\theta_i} g_1 fm + \frac{1}{2} g_3 fm + g_4 fs + \frac{1}{2} g_5 mr}.$$

As the g_i' s are polynomials with positive coefficients, **then condition (22) together with (23) permit us to choose** the sequence $\{\sigma_q\}_{q\in N}$ satisfying (17). This ends the proof of the Theorem. By application of the preliminary observations, we have the following,

Corollary 1 *Suppose that the reaction terms are continuously differentiable on R_+^4, then all positive solutions of (1)-(4) with initial data in $L^p(\Omega)$ are in $L^\infty(0,T_{max};L^p(\Omega))$ for all $p\geq 1$.*

Proof: If p is an integer, the proof is an immediate consequence of Theorem 3.1 and the trivial inequality

$$\int_\Omega (f(t,x)+m(t,x)+s(t,x))^p dx \leq C_1 L(t), \quad on \quad [0,T_{max}[, \tag{28}$$

$i = 0,1,\ldots,p,$

where C_1 is a positive constant depending on p.

If we suppose that the reaction terms are of polynomial growth

$$|g_i(f,m,s,r)| \leq C_2(f,m,s,r)[1+f+m+s+r]^l \quad on \quad R_+^3, \tag{29}$$

$i = 0,1,\ldots,p,$

where C_2 positive and bounded function on bounded subsets of R_+^4 we have the following

Proposition 2 *If the reaction terms are of polynomial growth with g_2, g_6, g_9 and g_{10} are sufficiently large, then all positive solutions of (1)-(4) with initial data in $L^\infty(\Omega)$ are global.*

Proof: From corollary 1, there exists a positive constant C_3 such that

$$\int_\Omega (1+f(t,x)+m(t,x)+s(t,x))^p dx \leq C_3, \quad on \quad [0,T_{max}[, \tag{30}$$

for all $p\geq 1$ and from (26) we have

$$|r_i(f,m,s,r)|^{\frac{p}{l+2}} \leq C_2(f,m,s,r)(1+f+m+s+r)^p, \quad on \quad [0,T_{max}[, \tag{31}$$

Since f,m,s and r are in $L^\infty([0,T^*[;L^p(\Omega))$, for all $p\geq 1$, then we can choose $p\geq 1$ such $\frac{p}{l+2} > \frac{N}{2}$ and from the preliminary observations the solution is global.

Remark 6 *The global existence can be proved under more general boundary conditions including homogeneous and nonhomogeneous Dirichlet, Neumann and mixed boundary conditions (see [28]). Also note, because the non linear semi-group S(t) in this case is regularizing [59], for initial data say $u_0 \in L^2(\Omega)$, for some $r>0$, $S(r)u_0 \in L^p(\Omega)$. We can now use the constructed functional (14) with initial data $S(r)u_0$, which is in $L^p(\Omega)$, so the local solution is in $L^p(\Omega)$, thus can't blow up and becomes global. Thus we have a priori $L^\infty(0,\infty;L^p(\Omega))$ bounds for data in $L^2(\Omega)$.*

V. BOUNDED ABSORBING SETS AND FURTHER A PRIORI ESTIMATES

5.1 Bounded absorbing sets

In this section we aim to investigate the asymptotic behavior of (1)-(4). We use the functional L_p to show the existence of bounded absorbing sets. Using the fact that the matrices B_{iq}, $i=\overline{0,q}$, $q=\overline{0,p}$ are positive definite, we can find a constant C_4 such that

$$I \geq p(p-1)C_4 \int_\Omega (df+m)^{p-2} |\nabla(f+m+s)|^2 dx, \tag{32}$$

and this gives

$$L_p'(t) + p(p-1)C_4\int_\Omega (f+m+s)^{p-2}|\nabla(f+m+s)|^2 dx \le 0, \quad on \quad [0,T_{\max}[, \tag{33}$$

by integrating (33) with respect to t, we deduce

$$\int_\Omega (f+m+s)^p dx + C_5\int_0^t\int_\Omega (f+m+s)^{p-2}|\nabla(f+m+s)|^2 dxds \le L_p(0), \quad on \quad [0,T_{\max}[, \tag{34}$$

then this inequality gives

$$f,m,s\in L^\infty\left([0,\infty[;L^p(\Omega)\right)\cap L^2\left([0,\infty[;H^1(\Omega)\right) \tag{35}$$

The above method shows the existence of bounded absorbing set in $L^p(\Omega)$, for all $p\ge 1$, and so in particular for $L^2(\Omega)$. Similar estimates are made in [48].

Remark 7 *Note, from (33) it is immediate that* $L_2'(t)<0$, *hence* $L_2(t)$ *is decreasing in time. From the form of the functional* $L_p(t)$ *in (14)-(15), it is clear that* $||f||_2^2$ *is also decreasing in time, and must enter some compact ball, by a finite time* t_1, *where* t_1 *will depend on the* $L^2(\Omega)$ *norm of the initial conditions, and the parameters in the system.*

For completeness we show certain details pertaining to the uniform $L^2(\Omega)$ estimates.

Let us begin by multiplying (1) by f, and integrating by parts over Ω, to obtain

$$\frac{1}{2}\frac{d}{dt}||f||_2^2+||\nabla f||_2^2=a_1\int_\Omega f^{\alpha_1+2}m^{\beta_1+1}dx-a_2\int_\Omega f^{\alpha_2+2}m^{\beta_2}dx. \tag{36}$$

We now use the positivity of f and m along with Holder's inequality to obtain

$$\frac{1}{2}\frac{d}{dt}||f||_2^2+||\nabla f||_2^2\le a_1||f||_{2(\alpha_1+2)}^{\alpha_1+2}||m||_{2(\beta_1+1)}^{\beta_1+1}\le C. \tag{37}$$

This follows via the a priori L^p bound on the solutions, and hence in particular for $p=\max(2(\alpha_1+2),2(\beta_1+1))$.

Note here C only depends on the $L^2(\Omega)$ norm of the initial data, and is independent of time. The C here comes from $L_2(0)$, which can be bounded by $C_1(||f_0||_2^2+||m_0||_2^2+||s_0||_2^2)$, where C_1 is a pure constant. Thus we use Poincare's inequality to obtain,

$$\frac{1}{2}\frac{d}{dt}||f||_2^2+||f||_2^2\le C_1(||f_0||_2^2+||m_0||_2^2+||s_0||_2^2). \tag{38}$$

Remark 8 *Typically, in order to make dissipative estimates, we require an inequality of the form* $\frac{d}{dt}||u||_V+C_2||u||_V\le C_1$, *for a state variable u in some function space V. The* C_1,C_2 *are pure constants, that could depend on the parameters in the problem, but not on the initial condition [50]. Methods in [50] show that typically if we choose* $t_1=\frac{1}{C_2}\ln(||u_0||_V)$, *then for times* $t>t_1$, *we have that* $||u||_V\le 1+\frac{C_1}{C_2}$. *In our setting the R.H.S does depend on the initial condition, however we can give an* (ε,δ) *argument to show that the* $L^2(\Omega)$ *norm of the solutions is still absorbed by a finite time* t_1. *Note in the estimates it is assumed conditions (20)-(23)), and conditions (16)-(18) hold.*

Via the use of Gronwall's lemma [50] in (38) we obtain

$$||f||_2^2\le e^{-2t}||f_0||_2^2+\frac{C_1}{2}(||f_0||_2^2+||m_0||_2^2+||s_0||_2^2)(1-e^{-2t}). \tag{39}$$

Note for any $2>\varepsilon>0$, there exists a $t=T^*(\varepsilon)$, s.t $\frac{e^{2t}-1}{e^{2t}}=e^{-\varepsilon t}$. Thus for $t\in[0,T^*(\varepsilon)]$, we have that $\frac{e^{2t}-1}{e^{2t}}\le e^{-\varepsilon t}$.

Case 1: $\frac{1}{\varepsilon}\ln(||f_0||_2^2+||m_0||_2^2+||s_0||_2^2)\le T^*(\varepsilon)$.

We assume $||f_0||_2^2+||m_0||_2^2+||s_0||_2^2>1$, else the absorbing set is trivial from (34).

Using the fact that $\frac{e^{2t}-1}{e^{2t}}\le e^{-\varepsilon t}$ for $t\in[0,T^*(\varepsilon)]$ $on \quad [0,T_{\max}[$, , in (39) we obtain,

$$||f||_2^2\le e^{-2t}||f_0||_2^2+\frac{C_1}{2}(||f_0||_2^2+||m_0||_2^2+||s_0||_2^2)e^{-\varepsilon t}. \tag{40}$$

Let's choose $t_0=\frac{1}{2}\ln(||f_0||_2^2)$. Also given (f_0,m_0,s_0) an $\varepsilon>0$, we can find a $\delta>0$ s.t

$$\frac{1}{\varepsilon}<\frac{1}{\delta}<\frac{T^*(\varepsilon)}{\ln(||f_0||_2^2+||m_0||_2^2+||s_0||_2^2)}. \tag{41}$$

Now we choose t_1^* such that

$$t_1^*=\frac{1}{\delta}\ln(||f_0||_2^2+||m_0||_2^2+||s_0||_2^2), \tag{42}$$

Note, given $2>\varepsilon>0$, we can always find $\delta>0$ via (41), so that $t_1^*<T^*(\varepsilon)$.

Finally we choose $t_1=\max(t_0,t_1^*)$, then we have

$$\|f\|_2^2 \le 1+\frac{C_1}{2},\ \forall\ t>t_1.$$

(43)

Case 2: $\frac{1}{\varepsilon}\ln(\|f_0\|_2^2+\|m_0\|_2^2+\|s_0\|_2^2)>T^*(\varepsilon)$,

This might be the situation if for example the given initial data (f_0,m_0,s_0) was very large. Then similarly as earlier we have $\|f\|_2^2 \le e^{-2t}\|f_0\|_2^2+\frac{C_1}{2}(\|f_0\|_2^2+\|m_0\|_2^2+\|s_0\|_2^2)e^{-\varepsilon t}$. (44)

Also given (f_0,m_0,s_0) an $\varepsilon>0$, we can find a $\delta>0$ s.t

$$\frac{1}{\delta}<\frac{T^*(\varepsilon)}{\ln(\|f_0\|_2^2+\|m_0\|_2^2+\|s_0\|_2^2)},\ 0<\delta<\varepsilon. \quad (45)$$

Thus

$$\|f\|_2^2 \le e^{-2t}\|f_0\|_2^2+\frac{C_1}{2}(\|f_0\|_2^2+\|m_0\|_2^2+\|s_0\|_2^2)e^{-\delta t}. \quad (46)$$

Now we choose

$$t_1^*=\frac{1}{\delta}\ln(\|f_0\|_2^2+\|m_0\|_2^2+\|s_0\|_2^2)), \quad (47)$$

Thus given $2>\varepsilon>0$, we can always find $\delta>0$ via (45), so that $t_1^*<T^*(\varepsilon)$.

Finally we choose $t_1=\max(t_0,t_1^*)$, and we have

$$\|f\|_2^2 \le 1+\frac{C_1}{2},\ \forall\ t>t_1. \quad (48)$$

We next demonstrate next the $H_0^1(\Omega)$ estimates with f. We integrate (37) in the time interval from $[t,t+1]$ for any $t\ge t_1$, to obtain

$$\|f(t+1)\|_2^2+\int_t^{t+1}\|\nabla f\|_2^2 ds \le C_5+\|f(t)\|_2^2 \le C_6,\ \forall t\ge t_1. \quad (49)$$

Remark 9 *Note the* C_6 *absorbs* C_5 *and* $1+\frac{C_1}{2}$ *from (48), so* $1+\frac{C_1}{2}+C_5<C_6$ *.*

Thus we have the following uniform integral in time bound

$$\int_t^{t+1}\|\nabla f\|_2^2 ds \le C_6,\ \forall t\ge t_1, \quad (50)$$

using the Mean Value Theorem for integrals, there exists $t^*\in[t,t+1]$ such that for all $t>t_1$, we obtain

$$\|\nabla f(t^*)\|_2^2 ds \le C_6, \quad (51)$$

We next multiply (1) by $-\Delta f$ and integrate by parts over Ω. For such higher order Sobolev estimates, we will assume f and Δf satisfy the same boundary conditions, and similarly the same is true for the other components. Thus we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla f\|_2^2+\|\Delta f\|_2^2 = a_1\int_\Omega f^{\alpha_1+1}m^{\beta_1+1}(-\Delta f)dx - a_2\int_\Omega f^{\alpha_2+1}m^{\beta_2}(-\Delta f)dx. \quad (52)$$

Then employing Young's inequality yields,

$$\frac{1}{2}\frac{d}{dt}\|\nabla f\|_2^2+\|\Delta f\|_2^2 \le \frac{1}{4}\|\Delta f\|_2^2+2\|f\|_{4(\alpha_1+1)}^{4(\alpha_1+1)}+2\|m\|_{4(\beta_1+1)}^{4(\beta_1+1)} + \frac{1}{4}\|\Delta f\|_2^2+2\|f\|_{4(\alpha_2+1)}^{4(\alpha_2+1)}+2\|m\|_{4(\beta_2)}^{4(\beta_2)},$$

which via the a priori L^p bounds on the solutions, hence in particular for $p=\max(4(\alpha_1+1),4(\beta_1+1),4(\alpha_2+1),4(\beta_2))$, leads to

$$\frac{d}{dt}\|\nabla f\|_2^2+\|\Delta f\|_2^2 \le 4\|f\|_{4(\alpha_1+1)}^{4(\alpha_1+1)}+4\|m\|_{4(\beta_1+1)}^{4(\beta_1+1)} +4\|f\|_{4(\alpha_2+1)}^{4(\alpha_2+1)}+4\|m\|_{4\beta_2}^{4\beta_2} \le C. \quad (53)$$

Now using the Sobolev embedding of $H^2(\Omega)\hookrightarrow H_0^1(\Omega)$, we obtain

$$\frac{d}{dt}\|\nabla f\|_2^2+C_1\|\nabla f\|_2^2 \le C.$$

(54)

Grönwall Lemma via integration in the time interval $[t^*,t]$ yields the following uniform bound

$$\|\nabla f\|_2^2 \le \frac{C}{C_1}\left(1-e^{-(t-t^*)}\right)+e^{-(t-t^*)}\|\nabla f(t^*)\|_2^2 \le \frac{C}{C_1}+C_6, \qquad \forall t\ge t^*\ge t_1. \quad (55)$$

This follows via (51).

Remark 10 *Note, (51) holds for any* $t>t^*$*. However, the reason we use* t^**, is to first derive a uniform in time bound on the* $\|\nabla f(t^*)\|_2^2$ *via (51), so that the* $e^{-(t-t^*)}\|\nabla f(t^*)\|_2^2$ *term can be absorbed, uniformly in time for times* $t\ge t^*\ge t_1$*, using (51).*

Remark 11 *The strategy for the uniform* $H_0^1(\Omega)$ *estimates for the m,s,r components is similar. That is we can derive a finite time* t_3^m *s.t.* $\|\nabla m\|_2^2\le C$*, for* $t>t_3^m$*. Here the finiteness of the time* t_3^m*, comes via the methods similar to (36)-(55), where we use the equation for m via (2). Similarly we can* *derive a finite time* t_3^s *s.t.* $\|\nabla s\|_2^2 \leq C$, *for* $t > t_3^s$, *and we can derive a finite time* t_3^r *s.t.* $\|\nabla r\|_2^2 \leq C$, *for* $t > t_3^r$.

This leads us to state the following Lemma.

Lemma 5.1. *Let f,m,s be solutions to (1)-(4) with* $(f_0, m_0, s_0, r_0) \in L^2(\Omega)$. *Assume conditions (20)-(23) and conditions (16)-(18) hold, and the finite* $H_0^1(\Omega)$ *absorption times for the components f, m, s, r are* t_1, t_3^m, t_3^s *and* t_3^r *respectively. We denote* t_3^* *by*

$$t_3^* = \max(t_1, t_3^m, t_3^s, t_3^r).$$

There exists a constant C independent of time and initial data, and depending only on the parameters in (1)-(4), such that for any $t > t_3^*$ *the following uniform a priori estimates hold:*

$$\|f\|_2^2 \leq C, \quad (56)$$
$$\|\nabla f\|_2^2 \leq C, \quad (57)$$
$$\|m\|_2^2 \leq C, \quad (58)$$
$$\|\nabla m\|_2^2 \leq C, \quad (59)$$
$$\|s\|_2^2 \leq C, \quad (60)$$
$$\|\nabla s\|_2^2 \leq C, \quad (61)$$
$$\|r\|_2^2 \leq C, \quad (62)$$
$$\|\nabla r\|_2^2 \leq C. \quad (63)$$

5.2 Local In Time a Priori Estimate for Δf.

Our goal now is to show that we can derive a priori $H^2(\Omega)$ bounds on the solutions to (1)-(4). To this end we next multiply (1) by $-\Delta f$ and integrate by parts over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla f\|_2^2 + d_1\|\Delta f\|_2^2 = a_1\int_\Omega f^{\alpha_1+1}m^{\beta_1+1}(-\Delta f)dx - a_2\int_\Omega f^{\alpha_2+1}m^{\beta_2}(-\Delta f)dx, \quad (64)$$

Then employing Young's inequality yields,

$$\frac{1}{2}\frac{d}{dt}\|\nabla f\|_2^2 + d_1\|\Delta f\|_2^2 \leq \frac{d_1}{4}\|\Delta f\|_2^2 + 2\|f\|_{4(\alpha_1+1)}^{4(\alpha_1+1)} + 2\|m\|_{4(\beta_1+1)}^{4(\beta_1+1)} + \frac{d_1}{4}\|\Delta f\|_2^2 + 2\|f\|_{4(\alpha_2+1)}^{4(\alpha_2+1)} + 2\|m\|_{4(\beta_2)}^{4(\beta_2)},$$

which via the a priori L^p bounds on the solutions, hence in particular for $p = \max(4(\alpha_1+1), 4(\beta_1+1), 4(\alpha_2+1), 4(\beta_2))$ leads to

$$\frac{d}{dt}\|\nabla f\|_2^2 + d_1\|\Delta f\|_2^2 \leq 4\left[\|f\|_{4(\alpha_1+1)}^{4(\alpha_1+1)} + \|m\|_{4(\beta_1+1)}^{4(\beta_1+1)} + \|f\|_{4(\alpha_2+1)}^{4(\alpha_2+1)} + \|m\|_{4\beta_2}^{4\beta_2}\right] \leq C. \quad (65)$$

Note the regularizing properties of the semigroup yield L^p (**p>2**) bounds on the solution, for initial data in L^2. We now integrate (65) above from t_3^* to t to obtain

$$\int_{t_3^*}^{t} \|\Delta f\|_2^2 ds \leq \int_{t_3^*}^{t} C ds.$$

In particular choosing

$$t = t_3^* + 1,$$

yields

$$\frac{1}{(t_3^*+1) - t_3^*}\int_{t_3^*}^{t_3^*+1} \|\Delta f\|_2^2 ds \leq C.$$

Hence using a mean value Theorem for integrals we obtain that there exists a time $t_3^{**} \in [t_3^*, t_3^*+1]$ such that the following estimate holds uniformly

$$\|\Delta f(t_3^{**})\|_2^2 \leq C.$$

Remark 12 *The strategy for the uniform integral in time* $H^2(\Omega)$ *estimates for the m,s,r components is similar. That is we can derive a finite time* t_{3*}^m *s.t.*

$$\int_{t_{3*}^m}^{t_{3*}^m+1} \|\Delta m\|_2^2 dt \leq C, \text{ for } t > t_{3*}^m.$$

Similarly we can derive a finite time t_{3*}^s *s.t.*

$$\int_{t_{3*}^s}^{t_{3*}^s+1} \|\Delta s\|_2^2 dt \leq C, \text{ for } t > t_{3*}^s,$$

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and we can derive a finite time t^r_{3*} *s.t.*

$$\int_{t^r_{3*}}^{t^r_{3*}+1} ||\Delta r||_2^2 dt \le C.$$

5.3 Uniform A Priori H^2 Estimate For f.

We multiply Equation (1) by $\Delta^2 f$ and integrate by parts over Ω to obtain

$$\frac{d\|\Delta f\|_2^2}{dt}+d_1\|\nabla(\Delta f)\|_2^2$$
$$\le\left(\int_\Omega \nabla(a_1 f^{\alpha_1+1} m^{\beta_1+1}-a_2 f^{\alpha_2+1} m^{\beta_2})\cdot\nabla(\Delta f)d\mathbf{x}\right)=$$
$$\int_\Omega (a_1 f^{\alpha_1+1}(\beta_1+1)m^{\beta_1}\nabla m+a_1 m^{\beta_1+1}(\alpha_1+1)f^{\alpha_1}\nabla f)\cdot\nabla(\Delta f)d\mathbf{x}$$
$$-\int_\Omega (a_2 f^{\alpha_2+1}(\beta_2)m^{\beta_2-1}\nabla m+a_2 m^{\beta_2}(\alpha_2+1)f^{\alpha_2}\nabla f)\cdot\nabla(\Delta f)d\mathbf{x}$$
$$\le C_2+\frac{d_1}{2}\|\nabla(\Delta f)\|_2^2+C_3\|\Delta f\|_2^2+C_4\|\Delta m\|_2^2. \qquad (66)$$

This follows via the a priori $L^p(\Omega)$ bounds on the solution, the embedding of $H^2(\Omega)\hookrightarrow W^{1,4}(\Omega)$, Cauchy-Schwartz and Young's inequalities. Now using the embedding of $H^3(\Omega)\hookrightarrow H^2(\Omega)$ we obtain,

$$\frac{d\|\Delta f\|_2^2}{dt}+\frac{d_1}{2}\|\Delta f\|_2^2\le C_2+C_3\|\Delta f\|_2^2+C_4\|\Delta m\|_2^2. \qquad (67)$$

Now we recall the Uniform Grönwall Lemma

Lemma 5.2. (Uniform Gronwall Lemma) *Let β,ζ and h be nonnegative functions in $L^1_{loc}[0,\infty[$. Assume that β is absolutely continuous on $]0,\infty[$ and the following differential inequality is satisfied.*

$$\frac{d\beta}{dt}\le\zeta\beta+h, \text{ for } t>0. \qquad (68)$$

If there exists a finite time $t_1>0$ and some $r>0$ such that

$$\int_t^{t+r}\zeta(\tau)d\tau\le A,\quad \int_t^{t+r}\beta(\tau)d\tau\le B \quad \text{and} \int_t^{t+r}h(\tau)d\tau\le C, \qquad (69)$$

for any $t>t_1$, where A, B and C are some positive constants, then

$$\beta(t)\le\left(\frac{B}{r}+C\right)e^A, \text{ for any } t>t_1+r. \qquad (70)$$

Thus using

$$\beta(\tau)=|\Delta f|_2^2,\ \zeta(\tau)=C_3-\frac{d_1}{2},\ h(\tau)=C_2+C_4|\Delta m|_2^2, \qquad (71)$$

and application of the above lemma yields

Lemma 5.3. *Let f,m,s be solutions to (1)-(4) with $(f_0,m_0,s_0,r_0)\in L^2(\Omega)$. Assume conditions (20)-(23) and conditions (16)-(18) hold, and we have the finite integral in time $H^2(\Omega)$ estimates for the components f, m, s, that is,*

$$\int_{t_3^*}^{t_3^*+1}||\Delta f||_2^2 dt\le C,\ \forall t>t_3^*$$
$$\int_{t^m_{3*}}^{t^m_{3*}+1}||\Delta m||_2^2 dt\le C,\ \forall t>t^m_{3*}$$
$$\int_{t^s_{3*}}^{t^s_{3*}+1}||\Delta s||_2^2 dt\le C\ \ \forall t>t^s_{3*} \qquad (66)$$

We denote t_4^* by

$$t_4^*=\max(t_3^*,t^m_{3*},t^s_{3*}).$$

Then there exists a constant C independent of time and initial data, and depending only on the parameters in (1)-(4), such that for any $t>t_4^*$ the following uniform a priori estimates hold:

$$||f(t)||_{H^2(\Omega)}\le C,\ \forall t\ge t_4^*+1,$$
$$||m(t)||_{H^2(\Omega)}\le C,\ \forall t\ge t_4^*+1,$$
$$||s(t)||_{H^2(\Omega)}\le C,\ \forall t\ge t_4^*+1.$$

Thus the existence of a bounded absorbing set in $H^2(\Omega)$ has also been established.

5.4. Uniform a Priori Estimates for $\frac{\partial f}{\partial t}$

From (1) via brute force we obtain

$$\left|\left|\frac{\partial f}{\partial t}\right|\right|_2^2$$
$$=\int_\Omega\left(d_1\Delta f+a_1 f^{\alpha_1+1}m^{\beta_1+1}-a_2 f^{\alpha_2+1}m^{\beta_2}\right)^2 d\mathbf{x}$$
$$\le C\|\Delta f\|_2^2+C_1\|f\|_{4(\alpha_1+1)}^{4(\alpha_1+1)}+C_2\|m\|_{4(\beta_1+1)}^{4(\beta_1+1)}$$

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$$\leq C\|\Delta f\|_2^2 + C_3\|\Delta f\|_2^2 + C_4\|\Delta m\|_2^2. \qquad (72)$$

This follows via the priori $L^p(\Omega)$ bounds on the solution, as well as the compact embedding of $H^2(\Omega)\hookrightarrow L^p(\Omega)$, $\forall p$ (since the spatial dimension $n\leq 3$). Similar estimates can be derived for the m,s components. We can now state the following Lemma,

Lemma 5.4. *Consider (1)-(4), for any solutions f,m,s and r of the system with* $(f_0,m_0,s_0,r_0)\in L^2(\Omega)$*. Assume conditions (20)-(23) and conditions (16)-(18) hold. We denote* t_6^* *by* $t_6^*=t_4^*+1$*.*
*Then there exists a constant **C**, independent of time and initial data such that the following estimates hold uniformly*

$$\left\|\frac{\partial f}{\partial t}\right\|_2^2 \leq C, \qquad \forall \geq t_6^*,$$

$$\left\|\frac{\partial m}{\partial t}\right\|_2^2 \leq C, \qquad \forall \geq t_6^*,$$

$$\left\|\frac{\partial s}{\partial t}\right\|_2^2 \leq C, \qquad \forall \geq t_6^*,$$

$$\left\|\frac{\partial r}{\partial t}\right\|_2^2 \leq C, \qquad \forall \geq t_6^*$$

This easily follows via the estimates in Lemma 5.5. We next make a local in time estimate on $\frac{\partial \nabla f}{\partial t}$ We take the partial derivative w.r.t t of (1) and multiply the resulting equation by $\frac{\partial f}{\partial t}$ and integrate by parts over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial f}{\partial t}\right\|_2^2 + d_1\left\|\frac{\partial \nabla f}{\partial t}\right\|_2^2 \leq$$

$$\int_\Omega \left(a_1(\beta_1+1)f^{\alpha_1+1}m^{\beta_1}\frac{\partial m}{\partial t} + a_1(\alpha_1+1)f^{\alpha_1}m^{\beta_1+1}\frac{\partial f}{\partial t}\right)\left(\frac{\partial f}{\partial t}\right)d\mathbf{x}$$

$$+\int_\Omega \left(-a_2(\beta_2)f^{\alpha_2+1}m^{\beta_2-1}\frac{\partial m}{\partial t} - a_2(\alpha_2+1)f^{\alpha_2}m^{\beta_2}\frac{\partial f}{\partial t}\right)\left(\frac{\partial f}{\partial t}\right)d\mathbf{x}$$

$$\leq$$

$$C\|f\|_\infty^{\alpha_1}\|m\|_\infty^{(\beta_1+1)}\left(\left\|\frac{\partial m}{\partial t}\right\|_2^2 + \left\|\frac{\partial f}{\partial t}\right\|_2^2\right) + C\|f\|_\infty^{\alpha_1}\|m\|_\infty^{(\beta_1+1)}\left(\left\|\frac{\partial f}{\partial t}\right\|_2^2\right)$$

$$+C\|f\|_\infty^{(\alpha_2+1)}\|m\|_\infty^{(\beta_2-1)}\left(\left\|\frac{\partial m}{\partial t}\right\|_2^2 + \left\|\frac{\partial f}{\partial t}\right\|_2^2\right) + C\|f\|_\infty^{\alpha_2}\|m\|_\infty^{\beta_2}\left(\left\|\frac{\partial f}{\partial t}\right\|_2^2\right). \qquad (72)$$

Thus integrating the above in the time interval $[t_6^*, t_6^*+1]$ yields

$$\int_{t_6^*}^{t_6^*+1}\left(\left\|\frac{\partial \nabla f}{\partial s}\right\|_2^2\right)ds \leq$$

$$C\|f\|_\infty^{\alpha_1}\|m\|_\infty^{(\beta_1+1)}\int_{t_6^*}^{t_6^*+1}\left(\left\|\frac{\partial m}{\partial s}\right\|_2^2 + \left\|\frac{\partial f}{\partial s}\right\|_2^2\right)ds + C\|f\|_\infty^{\alpha_1}\|m\|_\infty^{(\beta_1+1)}\int_{t_6^*}^{t_6^*+1}\left(\left\|\frac{\partial f}{\partial s}\right\|_2^2\right)ds$$

$$+C\|f\|_\infty^{(\alpha_2+1)}\|m\|_\infty^{(\beta_2-1)}\int_{t_6^*}^{t_6^*+1}\left(\left\|\frac{\partial m}{\partial s}\right\|_2^2 + \left\|\frac{\partial f}{\partial s}\right\|_2^2\right)ds + C\|f\|_\infty^{\alpha_2}\|m\|_\infty^{\beta_2}\int_{t_6^*}^{t_6^*+1}\left(\left\|\frac{\partial f}{\partial s}\right\|_2^2\right)ds.$$

$$\leq C\|f(t_6^*)\|_{H^2(\Omega)}^{\alpha_1}\|m(t_6^*)\|_{H^2(\Omega)}^{(\beta_1+1)}\int_{t_6^*}^{t_6^*+1}\left(\left\|\frac{\partial m}{\partial s}\right\|_2^2 + \left\|\frac{\partial f}{\partial s}\right\|_2^2\right)ds$$

$$+C\|f(t_6^*)\|_{H^2(\Omega)}^{\alpha_1}\|m(t_6^*)\|_{H^2(\Omega)}^{(\beta_1+1)}\int_{t_6^*}^{t_6^*+1}\left(\left\|\frac{\partial f}{\partial s}\right\|_2^2\right)ds$$

$$+C\|f(t_6^*)\|_{H^2(\Omega)}^{(\alpha_2+1)}\|m(t_6^*)\|_{H^2(\Omega)}^{(\beta_2-1)}\int_{t_6^*}^{t_6^*+1}\left(\left\|\frac{\partial m}{\partial s}\right\|_2^2 + \left\|\frac{\partial f}{\partial s}\right\|_2^2\right)ds$$

$$+C\|f(t_6^*)\|_{H^2(\Omega)}^{\alpha_2}\|m(t_6^*)\|_{H^2(\Omega)}^{\beta_2}\int_{t_6^*}^{t_6^*+1}\left(\left\|\frac{\partial f}{\partial s}\right\|_2^2\right)ds. \qquad (73)$$

$$\leq C.$$

This follows via the regularizing properties of the semigroup, Lemma 5.5, Lemma 5.4 and the embedding of $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$and hence using a mean value Theorem for integrals we obtain that there exists a time $t_6^{**}\in\left[t_6^*, t_6^*+1\right]$ such that the following estimate holds uniformly

$$\left\|\frac{\partial \nabla f(t_6^{**})}{\partial t}\right\|_2^2 \leq C.$$

We will next make a uniform in time estimate for $\left\|\frac{\partial \nabla f}{\partial t}\right\|_2^2$, where the previous estimate will be used. We take the time derivative of (1), then multiply through by $-\Delta\frac{\partial f}{\partial t}$ and integrate by parts over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\left|\frac{\partial \nabla f}{\partial t}\right|_2^2 + d_1\left|\Delta\left(\frac{\partial f}{\partial t}\right)\right|_2^2$$

$$\leq$$

$$a_1\left(\int_\Omega(\beta_1+1)f^{\alpha_1+1}m^{\beta_1}\frac{\partial m}{\partial t}+(\alpha_1+1)f^{\alpha_1}m^{\beta_1+1}\frac{\partial f}{\partial t}\right)\left(-\Delta\frac{\partial f}{\partial t}\right)dx$$

$$-a_2\left(\int_\Omega\beta_2 f^{\alpha_2+1}m^{\beta_2-1}\frac{\partial m}{\partial t}+(\alpha_2+1)f^{\alpha_2}m^{\beta_2}\frac{\partial f}{\partial t}\right)\left(-\Delta\frac{\partial f}{\partial t}\right)dx$$

$$\leq$$

$$C\max\left\{\|f\|_\infty^{2(\alpha_1+1)}\|m\|_\infty^{2(\beta_1+1)},\|f\|_\infty^{2(\alpha_2+1)}\|m\|_\infty^{2\beta_2}\right\}\left(\left\|\frac{\partial m}{\partial t}\right\|_2^2+\left\|\frac{\partial f}{\partial t}\right\|_2^2\right)$$

$$\frac{d_1}{2}\left\|\Delta\frac{\partial f}{\partial t}\right\|_2^2.$$

This follows from the product rule for differentiation, Cauchy-Schwartz inequality and the Sobolev embedding of $H_0^1(\Omega)\hookrightarrow L^4(\Omega)$. Now using the embedding of $H^2(\Omega)\hookrightarrow H_0^1(\Omega)$ we obtain

$$\frac{d}{dt}\left\|\frac{\partial\nabla f}{\partial t}\right\|_2^2+d_1\left\|\frac{\partial\nabla f}{\partial t}\right\|_2^2$$

$$\leq C\|f\|_{H^2}^K\|m\|_{H^2}^K\left(\left\|\frac{\partial f}{\partial t}\right\|_2^2+\left\|\frac{\partial m}{\partial t}\right\|_2^2\right)$$

$$\leq C.$$

Here $K=\max(2(\alpha_1+1),2(\beta_1+1),2(\alpha_2+1),2(\beta_2))$.

Thus via time integration in the interval $[t_6^{**},t]$ in the Grönwall Lemma we obtain

$$\left\|\frac{\partial\nabla f}{\partial t}\right\|_2^2\leq e^{-d_1(t-t_6^{**})}\left\|\frac{\partial\nabla f\left(t_6^{**}\right)}{\partial t}\right\|_2^2+\frac{1-e^{-d_1(t-t_6^{**})}}{d_1}C\leq C. \qquad (74)$$

$\forall t\geq t_6^{**}$.

We can make similar estimates for the other components, and derive similarly absorbing times $t_6^{**m},t_6^{**s},t_6^{**r}$, where $(t_6^{**m},t_6^{**s},t_6^{**r})$ are the absorption times for $\left\|\frac{\partial\nabla m(t)}{\partial t}\right\|_{L^2(\Omega)}$, $\left\|\frac{\partial\nabla r(t)}{\partial t}\right\|_{L^2(\Omega)}$, $\left\|\frac{\partial\nabla s(t)}{\partial t}\right\|_{L^2(\Omega)}$.

We thus state the following result,

Lemma 5.5. *Consider (1)-(5). For any solutions u,v,w,z to the system, there exists a constant* ***C*** *independent of time and initial data, and a time* $t_6^{***}=\max(t_6^{**},t_6^{**m},t_6^{**s},t_6^{**r})$, *such that the following estimates hold uniformly,*

$$\left\|\frac{\partial\nabla f}{\partial t}\right\|_{L^2(\Omega)}^2\leq C,\ \forall t\geq t_6^{***},$$

$$\left\|\frac{\partial\nabla m}{\partial t}\right\|_{L^2(\Omega)}^2\leq C,\ \forall t\geq t_6^{***},$$

$$\left\|\frac{\partial\nabla r}{\partial t}\right\|_{L^2(\Omega)}^2\leq C,\ \forall t\geq t_6^{***},$$

$$\left\|\frac{\partial\nabla s}{\partial t}\right\|_{L^2(\Omega)}^2\leq C,\ \forall t\geq t_6^{***}.$$

VI. EXISTENCE OF GLOBAL ATTRACTOR

In this section we prove the existence of a compact global attractor for system (1)-(4).

6.1 Preliminaries

Recall the phase space H introduced earlier

$H=L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega)$.

Also recall

$Y=H_0^1(\Omega)\times H_0^1(\Omega)\times H_0^1(\Omega)\times H_0^1(\Omega)$,

and

$$X=\left(H^2(\Omega)\cap H_0^1(\Omega)\right)\times\left(H^2(\Omega)\cap H_0^1(\Omega)\right)\times\left(H^2(\Omega)\cap H_0^1(\Omega)\right)\times\left(H^2(\Omega)\cap H_0^1(\Omega)\right)$$

Recall the following definitions

Definition 6.1. *Let $\mathcal{A}\subset H^2(\Omega)$, then $\mathcal{A}$ is said to be a (H,X) global attractor if the following conditions are satisfied*
i) $\mathcal{A}$ is compact in X.
ii) $\mathcal{A}$ is invariant, i.e., $S(t)\mathcal{A}=\mathcal{A}$, $t\geq 0$.
iii) If B is bounded in H, then

$$dist_X(S(t)B,\mathcal{A})\to 0,\ t\to\infty.$$

Definition 6.2. (Asymptotic compactness) *The semi-group $\{S(t)\}_{t\geq 0}$ associated with a dynamical system is said to be asymptotically compact in $H^2(\Omega)$ if for any $\{f_{0,n}\}_{n=1}^{\infty}$ bounded in $L^2(\Omega)$ and a sequence of times $\{t_n\to\infty\}$, $S(t_n)f_{0,n}$ possesses a convergent subsequence in $H^2(\Omega)$.*

Definition 6.3. (Bounded absorbing set) *A bounded set $\mathcal{B}$ in a reflexive Banach space H is called a bounded absorbing set if for each bounded subset U of H, there is a time T=T(U) depending on U, such that $S(t)U\subset\mathcal{B}$ for all*

*$t>T$. The number $T=T(U)$ is referred to as the **compactification time** for $S(t)U$. This is the time after which the semigroup compactifies.*

Also recall that if $\mathcal{A}$ is an (H,H) attractor, then in order to prove that it is a an (H,X) attractor it suffices to show the existence of a bounded absorbing set in X as well as demonstrate the asymptotic compactness of the semi-group in X, see [60]. We first state the following Lemma.

Lemma 6.4. *Consider the system described via, (20)-(23). Under conditions (1)-(4), there exists a (H,H) global attractor $\mathcal{A}$ for this system which is compact and invariant in H, and attracts all bounded subsets of H in the H metric.*

Proof: The existence of bounded absorbing set in H follow via the estimates derived in Lemma5.1. Furthermore the compact Sobolev embedding of

$$Y\hookrightarrow H$$

yields the asymptotic compactness of the semi-group $\{S(t)\}_{t\geq 0}$ in H. The existence of an (H,H) global attractor now follows.

6.2. One Point Attractor

In this subsection, we shall prove that the attractor is a one point attractor. We begin by the following

Proposition 3 *The unique fixed point of the semi group $\{S(t)\}_{t\geq 0}$ associated with the dynamical system (1)- (4) is the null solution, if the reaction term in the equation for r is not positive and (20)-(23) hold.*

Proof: Suppose that $\overline{u}=(\overline{v},\overline{r})$ is a fixed point of the semi group $\{S(t)\}_{t\geq 0}$, where $\overline{v}=(\overline{f},\overline{m},\overline{s})$.

First by supposing the reaction term in the equation for r is not positive, we can deduce easily that $\overline{r}\equiv 0$. for the other components we use the Lyapunov functional L_p: from equation (14)-(15).

Since for $\tau>0$, we have $S(t)\overline{v}=0$ on the interval $(0,\tau)$. This gives

$$L_p'(t)=:I+J=0,\ for all t\in(0,\tau)$$

Since $I\leq 0$ and $J\leq 0$ on the interval $(0,\tau)$, then $I=J\equiv 0, \forall t\in(0,\tau)$ Since each of the quadratic forms (with respect to ∇f, ∇m and ∇s) associated with the matrices B_{iq} $0\leq q\leq p-2$, $0\leq i\leq q$, given by (26) and appearing in the expression of the integral I given by (25) are positive. This gives

$$\nabla f=\nabla m=\nabla s=0, \forall t\in(0,\tau)$$

which means that $(\overline{f},\overline{m},\overline{s})$ is a constant vector. Using the homogenous Dirichlet boundary conditions, we deduce $\overline{f}=\overline{m}=\overline{s}=0$, on the interval $(0,\tau)$. This end the proof of the proposition.

Now we can state the following

Theorem 6.5 *The attractor of the semi group $\{S(t)\}_{t\geq 0}$ is a one point attractor.*

Proof: By direct application of A.V. Babin and M.I. Vishik [3] [Theorem 10.2, page 2.4], we can deduce the statement of the Theorem.

We next place sufficient conditions on the g_i and show that in certain special cases, f will decay exponentially to 0, via the following Lemma.

Lemma 6.6. *Consider the model system (1)- (4). If the reaction term in the equation for r is not positive, then for any constant g_i, $i=1,3,4,5,7,8,9$, and $f_0,m_0\in L^2(\Omega)$, we can choose g_2,g_6, s.t $f,m\to 0$, exponentially in the $L^2(\Omega)$ norm.*

Proof: If the reaction term on r is not positive, r trivially goes extinct leading to the extinction of s. This reduces (1)- (4) to

$$\partial_t f=d_1\Delta f+g_1 fm-g_2(f,m)f, \tag{75}$$

$$\partial_t m=d_2\Delta m+g_3 fm-g_6(f,m)m, \tag{76}$$

with Dirichlet boundary conditions. For the sake of simplicity let us assume $g_2(f,m)=C_1f^2$, $g_6(f,m)=C_2m^2$,

where C_1,C_2, will be chosen later. Then multiplying (75) by f and (76) by m and integrating by parts over Ω yields

$$\frac{1}{2}\frac{d}{dt}\|f\|_2^2+d_1\|\nabla f\|_2^2+C_1\|f\|_4^4=g_1\int_\Omega f^2 m dx, \tag{77}$$

$$\frac{1}{2}\frac{d}{dt}\|m\|_2^2+d_2\|\nabla m\|_2^2+C_2\|m\|_4^4=g_3\int_\Omega m^2 f dx. \tag{78}$$

Using Poincare's and Young's inequality with ε we obtain

$$\frac{1}{2}\frac{d}{dt}\|f\|_2^2+C_3\|f\|_2^2+C_1\|f\|_4^4\leq\frac{g_1}{2\varepsilon^2}\|f\|_4^4+\frac{g_1\varepsilon^2}{2}\|m\|_2^2, \tag{79}$$

$$\frac{1}{2}\frac{d}{dt}\|m\|_2^2+C_4\|m\|_2^2+C_2\|m\|_4^4\leq\frac{g_3}{2\varepsilon^2}\|m\|_4^4+\frac{g_3\varepsilon^2}{2}\|f\|_2^2. \tag{80}$$

Choosing ε small enough that is $\varepsilon<\min(\sqrt{\frac{C_3}{g_1}},\sqrt{\frac{C_4}{g_3}})$, and C_1,C_2 large enough that is $C_1>\left(\frac{g_1}{2}\right)\left(\frac{1}{\min(\frac{C_3}{g_1},\frac{C_4}{g_3})}\right)$,

we add up the above to obtain

$$\frac{1}{2}\frac{d}{dt}\|f\|_2^2+\frac{1}{2}\frac{d}{dt}\|m\|_2^2+C_5\|f\|_2^2+C_6\|m\|_2^2\leq 0. \quad (81)$$

Defining $V=m+f$, and $C_7=\min(C_5,C_6)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|V\|_2^2+C_7\|V\|_2^2\leq 0. \quad (82)$$

This yields

$$\|V\|_2^2\leq e^{-2C_7t}\|V_0\|_2^2, \quad (83)$$

thus we have

$$\lim_{t\to\infty}\|V\|_2^2\to 0, \quad (84)$$

which implies

$$\lim_{t\to\infty}\|f\|_2^2\to 0,\ \lim_{t\to\infty}\|m\|_2^2\to 0 \quad (85)$$

exponentially. This proves the Lemma.

Remark 13 *Note we must choose g_2,g_6 super linear at the very least, because choosing them as constant or sub linear can lead to blow-up in finite time for sufficiently large initial data, and then no convergence to equilibrium (or extinction state) is guaranteed [30].*

Remark 14. *In the special case that r goes extinct, which happens if the reaction term in (4) is non positive, s follows suit trivially. Then we are essentially left with a 2 species system for f,m. Special cases of this are tackled in [49] See pg.133, example 1.10 and references therein. Although under certain restrictions on the reaction terms existence of a global attractor can be proved (via the Simon-Lojasiewicz type techniques), convergence to equilibrium is another matter. For example, let us (assuming $r,s\to 0$) choose $g_1=2C_1,g_2=C_2f-C_3,g_3=2,g_6=m$. Then upon analyzing the Jacobian we see that if we choose $C_1=\frac{5}{8},C_2=\frac{1}{8}$ then $C_4=\frac{1}{2}$ (where $C_4=C_1-C_2$), what we obtain is $J_{11}|_{(f^*,m^*)}=\frac{3C_3}{2}$, while $J_{22}|_{(f^*,m^*)}=\frac{-C_3}{C_4}$. Since all constants are positive this says $J_{11}J_{22}<0$, and standard pattern formation results [64] tell us that there exist diffusion coefficients d_1,d_2 for which Turing instability will occur. Thus the base equilibrium state is driven unstable because of diffusion, and one will* not *have convergence to the spatially homogenous equilibrium solution.*

6.3. Asymptotic Compactness of the Semi-ggroup In X

In this section we demonstrate the asymptotic compactness property. We show calculations for *f*, the other variables follow similarly. Showing asymptotic compactness in $H^2(\Omega)$, would involve making uniform $H^3(\Omega)$ estimates and then using the Sobolev embedding of $H^3(\Omega)\hookrightarrow H^2(\Omega)$. This will be quite cumbersome, and so is circumvented altogether via the following strategy. We rewrite (1) as

$$d_1\Delta f=\frac{\partial f}{\partial t}+a_1f^{\alpha_1+1}m^{\beta_1+1}-a_2f^{\alpha_2+1}m^{\beta_2}. \quad (86)$$

We will demonstrate that every term on the right hand side of (86) is uniformly bounded in $L^2(\Omega)$. Thus we obtain that Δf is uniformly bounded in $L^2(\Omega)$, which will imply via elliptic regularity the uniform boundedness of f in $H^2(\Omega)$. Since this can be done for the other variables, the asymptotic compactness in X follows. To demonstrate this we state the following Lemma

Lemma 6.7. *The semi-group $\{S(t)\}_{t\geq 0}$ associated with the dynamical system (1)- (4) is asymptotically compact in X.*

Proof: Let us denote $f_n(t)=S(t)f_{0,n}$ and $u(t_n)=\frac{\partial f_n}{\partial t}|_{t=t_n}$. We have that

Error!

Via Lemma 5.5 we have for $t\geq t_6^{***}$

$$\left|\nabla\frac{\partial f}{\partial t}\right|_2\leq C.$$

Hence for n large enough $t_n\geq t_6^{***}$ and we obtain

$$\left|\nabla\frac{\partial f_n}{\partial t}\right|_2|_{t=t_n}\leq C.$$

Also via Lemma 5.1 we have the estimate

$$\|\nabla f\|_2\leq C.$$

Hence for n large enough $t_n\geq t_6^{***}$ and we obtain

$$\|\nabla f_n\|_2\leq C.$$

These uniform bounds allow us to extract weakly convergent subsequences. Thus we obtain

$u_n(t_n)\to u$ weakly in $H_0^1(\Omega)$.

$f_n(t_n)\to f$ weakly in $H_0^1(\Omega)$.

Now it trivially follows from the form of the reaction terms, and the simple algebraic inequality

$$\|F_n(f_n,m_n)-F(f,m)\|\leq C\left(\|f_n-f\|_2+\|m_n-m\|_2\right).$$

Here

$$F_n(f_n,m_n)=a_1 f_n^{\alpha_1+1}(t_n)m_n^{\beta_1+1}(t_n)-a_2 f_n^{\alpha_2+1}(t_n)m_n^{\beta_2}(t_n)$$

Thus from the classical functional analysis theory, see [], and the compact embedding of

$$H_0^1(\Omega)\hookrightarrow L^2(\Omega),$$

we obtain

$$u_n(t_n)\to u \text{ strongly in } L^2(\Omega),$$

$$f_n(t_n)\to f \text{ strongly in } L^2(\Omega),$$

$$F_n(f_n)\to F(f) \text{ strongly in } L^2(\Omega).$$

Using these convergent subsequences we obtain

$$\Delta f_n\to\Delta f \text{ strongly in } L^2(\Omega).$$

However this implies via elliptic regularity that

$$f_n\to f \text{ strongly in } H^2(\Omega).$$

This proves the Lemma.

We can now state the following result

Theorem 6.8. *Consider the reaction diffusion system described via (1)-(4). Under the conditions (20)-(23), there exists a (H,X) global attractor $\mathcal{A}$ for this system which is compact and invariant in X and attracts all bounded subsets of H in the X metric.*

Proof: The system is well posed via proposition 2, hence there exists a well defined semi-group $\{S(t)\}_{t\geq 0}$ for initial data in $L^2(\Omega)$. We already have the existence of an (H,H) global attractor via lemma 6.4. The estimates derived via Lemma 6.5 give us the existence of bounded absorbing sets in X. Lemma 6.7 proves the asymptotic compactness of the semi-group $\{S(t)\}_{t\geq 0}$ for the dynamical system associated with (1)-(4), in X. These results in conjunction prove the Theorem.

Remark 15 *Via standard methods [60] we can provide upper bounds on the Hausdorff and fractal dimensions of the global attractor in terms of parameters in the model. To derive these estimates we consider a volume element in the phase space, and try and derive conditions that will cause it to decay, as time goes forward. This enables an explicit upper bound for the Hausdorff dimension of the attractor*

$$d_H(A)\leq\left(\frac{C(a_i,b_i,\alpha_i,\beta_i)}{K_1}\right)^{\frac{3}{2}}|\Omega|+1, \qquad (87)$$

Numerical simulations show that this attractor is a one point attractor, see section 7.

VII. NUMERICAL SIMULATIONS

7.1 The basic model

We now provide the results of numerical simulations on (88) -(91). In order to demonstrate the proposed strategy we simulate (88) -(91), under a varied choice of parameters, and function *g*. When *g*=1, we have the TYC model, without a logistic control term. When $g=(1-(f+m+r+s)/K)$ the classical TYC model [43] is recovered.

$$\partial_t f=d_1\Delta f+a_1 fmg-b_1 f, \qquad (88)$$

$$\partial_t m=d_2\Delta m+a_2 fmg+b_2 fsg+c_2 mrg-e_2 m, \qquad (89)$$

$$\partial_t s=d_3\Delta s+a_3 mrg+b_3 rsg-e_3 s, \qquad (90)$$

$$\partial_t r=d_4\Delta r+\mu-b_4 r. \qquad (91)$$

In the simulations $\Omega=[0,\pi]$, so we are in a 1d spatial domain. We prescribe Dirichlet boundary conditions. The system is simulated in MATLAB R2014, using the PDE solver PDEPE. We experiment with various parameters, to obtain the spatio-temporal profiles of the solutions.

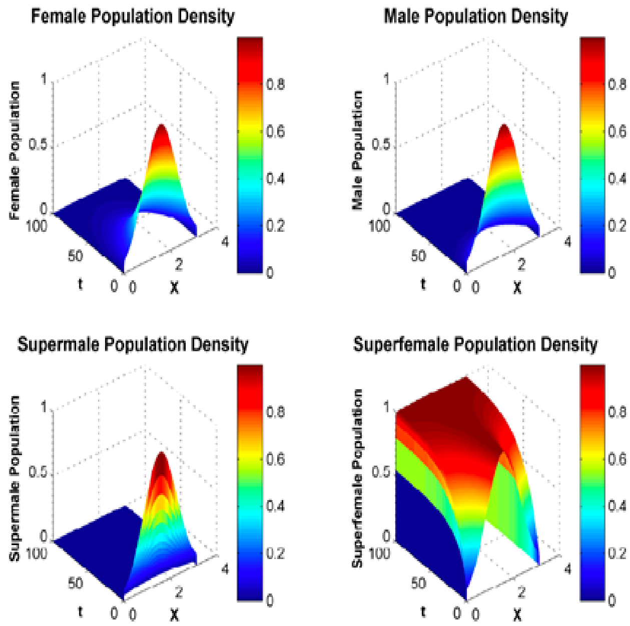

Figure 1: We fix $x=\pi/2$ and follow a trajectory in time for (88) -(91). The blue is the true trajectory, compared to $e^{-(0.012)t}$ in green. The clear exponential attraction of normal females to the extinction state is observed. The parameters are $d_1 = d_2 = d_3 = d4 = 0{:}001$; $a_1 = a_2 = 0{:}002$; $a_3 = b_1 = b_2 = b_3 = e_2 = 0{:}001$; $b_4=0{:}05$; $e_3 = 0{:}03$; $\mu = 0{:}5$. The initial data is taken to be $e^{-(x-\pi)^2}$.

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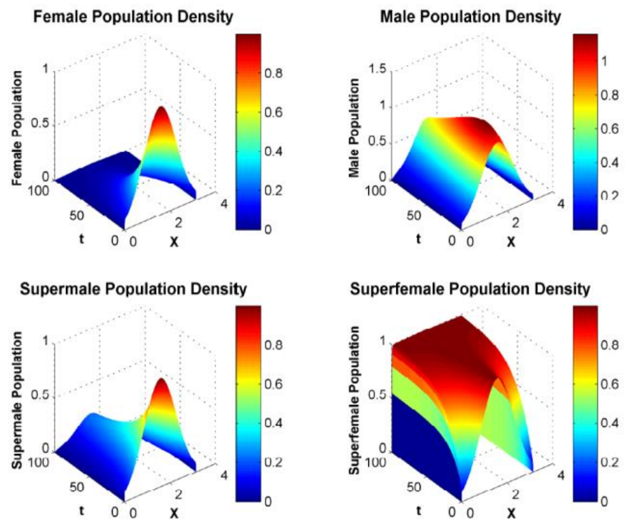


Figure 2: Here we consider a superlinear source term $\mu=\mu(r)=\mu r^2$. In this case r, will blow up in finite time. We investigate if this source term can cause a faster decay in the female species, in comparison to a constant μ perse. Surprisingly this is not so. We fix a spatial location and look at the decay of the trajectory of the female f. Here we are comparing (88) -(91) with g=1 to (88) -(91) with g=1 and $\mu=\mu r^2$, and same parameter set as in Fig.1 We observe that there is a sharper decay in f with a constant μ, than with $\mu=\mu r^2$.

7.2 Optimal Control

Motivated by the results in Fig. 1- 2 we consider the following ODE version of (88) -(91),

$$\frac{\partial f}{\partial t}=a_1fmg-b_1f,$$
$$\frac{\partial m}{\partial t}=a_2fmg+b_2fsg+c_2mrg-e_2m,$$
$$\frac{\partial s}{\partial t}=a_3mrg+b_3rsg-e_3s,$$
$$\frac{\partial r}{\partial t}=\mu(r)-b_4r \tag{92}$$

We want to compare the following 3 cases:

1. Case 1. $g=1,\mu(r)=\mu$;
2. Case 2. $g=1-\frac{f+m+s+r}{K},\mu(r)=\mu$;
3. Case 3. $g=1,\mu(r)=\mu r^2$.

We will use optimal control theory to illustrate which strategy is better for the eradication of wild females and wilde males. Here, consider the control problem

$$J(\mu)=max\int_0^T-(f+m)-\frac{1}{2}\mu^2dt. \tag{93}$$

We search for the optimal controls in the set U where

$$U=\{\mu|\mu \text{ measurable}, 0\le\mu<\infty, t\in[0,T], \forall T\}. \tag{94}$$

The goal is to seek an optimal μ^* s.t.,

$$J(\mu^*)=\mu\max\int_0^T-(f+m)-\frac{1}{2}\mu^2dt \tag{95}$$

We use the Pontryagin's maximum princinple to derive the necessary conditions on the optimal control. The Hamiltonian for J is given by

$$H=-(f+m)-\frac{1}{2}\mu^2+\lambda_1f'+\lambda_2m'+\lambda_3s'+\lambda_4r'. \tag{96}$$

We use the Hamiltonian to find a differential equation of the adjoint λ_i, i=1,2,3,4.

7.3 g=1,μ(r)=μ.

$$\lambda'_1(t)=1-\lambda_1(ma_1-b_1)-\lambda_2(ma_2+sb_2),$$
$$\lambda'_2(t)=1-\lambda_1a_1f-\lambda_2(fa_2+rc_2-e_2)-\lambda_3ra_3,$$
$$\lambda'_3(t)=-\lambda_2b_2f-\lambda_3(rb_3+e_3), \tag{97}$$
$$\lambda'_4(t)=-\lambda_2c_2m-\lambda_3(ma_3+sb_3)+\lambda_4b_4,$$

with the transversality condition gives as

$$\lambda_1(T)=\lambda_2(T)=\lambda_3(T)=\lambda_4(T)=0 \tag{98}$$

Now considering the optimality conditions, the Hamiltonian function is differentiated with respect to control variable μ resulting in

$$\frac{\partial H}{\partial \mu}=\lambda_4-\mu \tag{99}$$

Then a compact way of writing the optimal control μ is

$$\mu^*(t)=max(0,\lambda_4) \tag{100}$$

(92)

Theorem 7.1 *An optimal control $\mu^*\in U$ for the system (92) with $g=1,\mu(r)=\mu$ that maximizes the objective functional J is characterized by (100).*

7.4. $g=1-\frac{f+m+s+r}{K}$, μ(r)=μ;

$$\lambda'_1(t)=1\text{-}\lambda_1\left(gma_1-\frac{fma_1}{K}\text{-b}_1\right)\text{-}\lambda_2\left(gma_2-\frac{fma_2}{K}+\text{gsb}_2-\frac{mrc_2}{K}\right)$$
$$+\lambda_3\left(gma_2-\frac{mra_3}{K}+\frac{rsb_3}{K}\right), \quad (101.a)$$
$$\lambda'_2(t)=1\text{-}\lambda_1\left(fga_1-\frac{fma_1}{K}\right)\text{-}\lambda_2\left(-\frac{fma_2}{K}+fga_2-\frac{fsb_2}{K}-\frac{mrc_2}{K}+\text{grc}_2-e_2\right)$$
$$\text{-}\lambda_3\left(-\frac{mra_3}{K}+gra_3-\frac{rsb_3}{K}\right),$$

$$\lambda'_3(t)=\lambda_1\frac{fma_1}{K}+\lambda_2\left(\frac{fma_2}{K}+\frac{fsb_2}{K}-fgb_2+\frac{mrc_2}{K}\right)$$
$$+\lambda_3\left(\frac{mra_3}{K}+\frac{rsb_3}{K}-grb_3+e_3\right), \quad (101.b)$$
$$\lambda'_4(t)=\lambda_1\frac{fma_1}{K}+\lambda_2\left(-\frac{fma_2}{K}-\frac{fsb_2}{K}-\frac{mrc_2}{K}+gmc_2\right)$$
$$-\lambda_3\left(-\frac{mra_3}{K}+gma_3-\frac{rsb_3}{K}+gsb_3\right)+\lambda_4 b_4$$

with the transversality condition gives as

$$\lambda_1(T)=\lambda_2(T)=\lambda_3(T)=\lambda_4(T)=0 \quad (102)$$

Now considering the optimality conditions, the Hamiltonian function is differentiated with respect to control variable μ resulting in

$$\frac{\partial H}{\partial \mu}=\lambda_4-\mu \quad (103)$$

Then a compact way of writing the optimal control μ is

$$\mu^*(t)=max(0,\lambda_4) \quad (104)$$

Theorem 7.2 *An optimal control $\mu^*\in U$ for the system (9292) with $g=1-\frac{f+m+s+r}{K}$, $\mu(r)=\mu$ that maximizes the objective functional J is characterized by (104).*

7.5 g=1,μ(r)=μr²

$$\lambda'_1(t)=1\text{-}\lambda_1(ma_1-\text{b}_1)\text{-}\lambda_2(ma_2+\text{sb}_2),$$
$$\lambda'_2(t)=1\text{-}\lambda_1 a_1 f\text{-}\lambda_2(fa_2+\text{rc}_2-e_2)\text{-}\lambda_3\text{ra}_3,$$
$$\lambda'_3(t)=\text{-}\lambda_2\text{b}_2 f\text{-}\lambda_3(rb_3+\text{e}_3), \quad (105)$$
$$\lambda'_4(t)=\text{-}\lambda_2 c_2\text{m-}\lambda_3(ma_3+\text{sb}_3)+\lambda_4 b_4,$$

with the transversality condition gives as

$$\lambda_1(T)=\lambda_2(T)=\lambda_3(T)=\lambda_4(T)=0 \quad (106)$$

Now considering the optimality conditions, the Hamiltonian function is differentiated with respect to control variable μ resulting in

$$\frac{\partial H}{\partial \mu}=r^2\lambda_4-\mu \quad (107)$$

Then a compact way of writing the optimal control μ is

$$\mu^*(t)=max(0,r^2\lambda_4) \quad (108)$$

Theorem 7.3 *An optimal control $\mu^*\in U$ for the system (92) with $g=1,\mu(r)=\mu r^2$ that maximizes the objective functional J is charcterized by (108).*

7.6 Numerical Results for Optimal Control

In this subsection, we numerically simulate the model (92) with the 3 different cases. The following parameters will be used in the simulation,

$a_1=a_2=a_3=b_2=c_2=0.0045; b_2=b_3=0.009; b_4=e_2=e_3=0.12.$

Firstly, we set the initial population as follows

$$f_0=20, m_0=20, s_0=3, r_0=1. \quad (109)$$

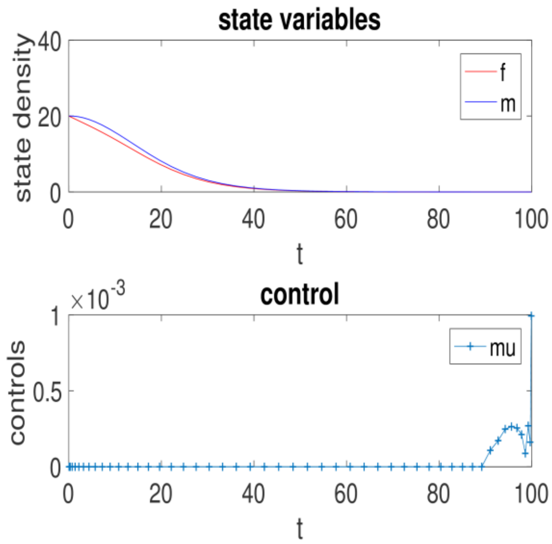


Fig 3:- Here, we simulate case 1 where g=1 and □(r)=□.

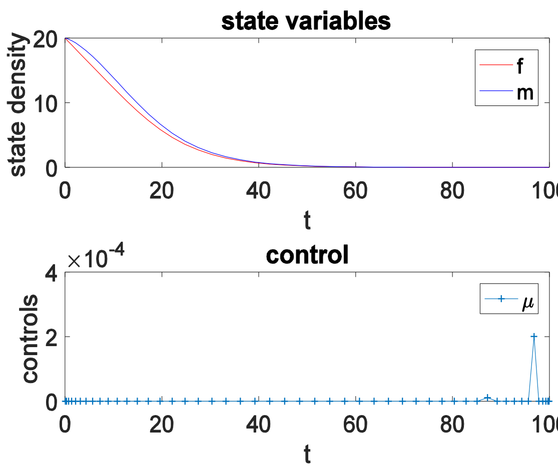


Fig 4:- Here, we simulate case 2 where $g=1-\frac{f+m+s+r}{K}$ and $\mu(r)=\mu$.

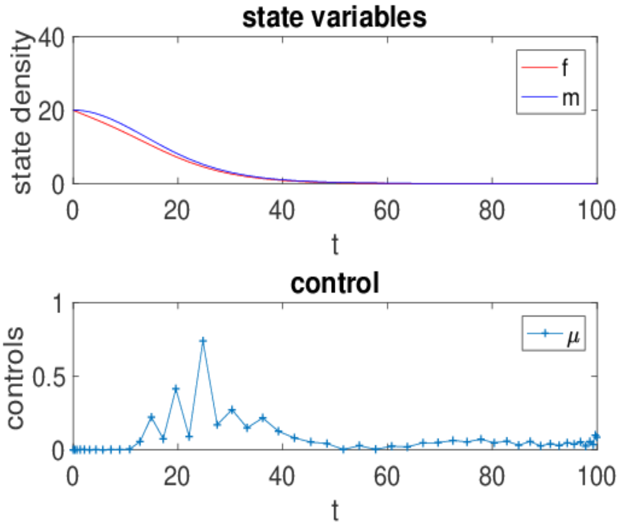
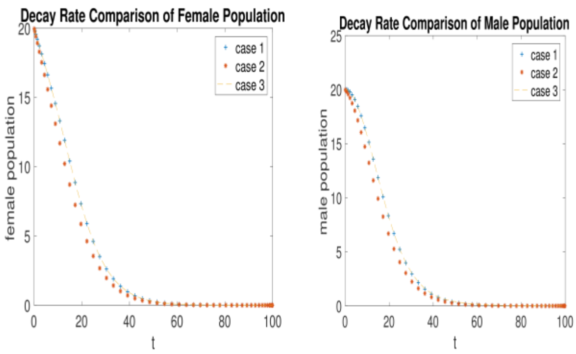


Fig 5:- Here, we simulate case 3 where g=1 and $\mu(r)=\mu r^2$.



A B

Figure 6: We take the initial conditions as $f_0=20, m_0=20, s_0=3, r_0=1$ and compare the decay rate of both females and males in each case.

Then we simulate the system with larger initial conditions,

$$f_0=90, m_0=90, s_0=8, r_0=6. \quad (110)$$

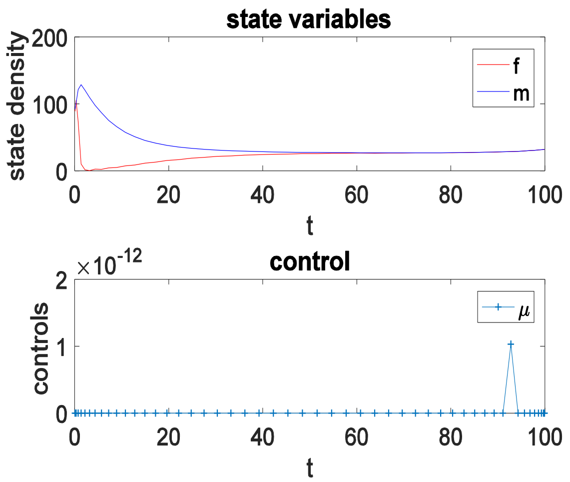


Fig 7:- Here, we simulate case 1 where g=1 and $\mu(r)=\mu$.

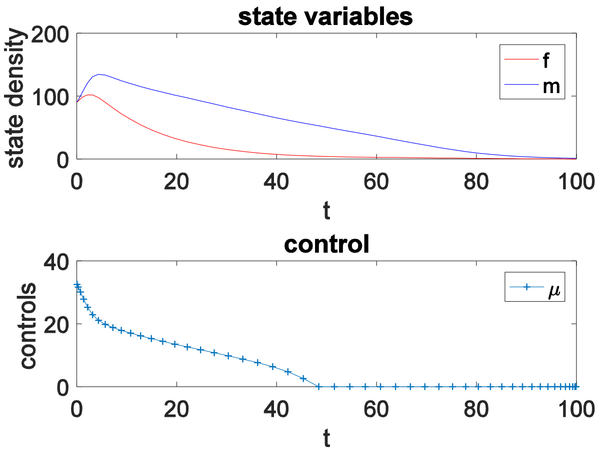


Fig 8: Here, we simulate case 2 where $g=1-\frac{f+m+s+r}{K}$ and $\mu(r)=\mu$.

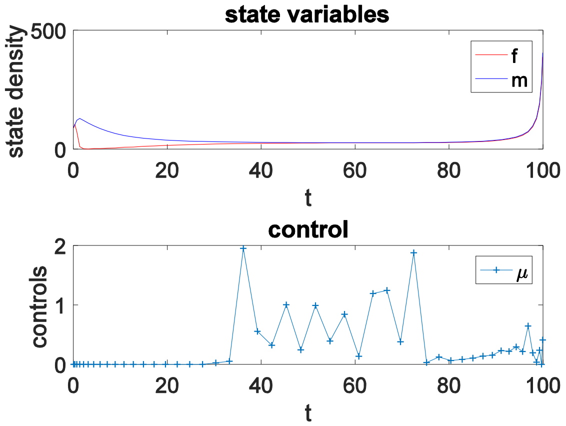


Fig 9:- Here, we simulate case 3 where g=1 and $\mu(r)=\mu r^2$.

For small initial population, cases 1-3 are all effective to eradicate wild females and males; however, case 3 requires larger μ for a certain period which is shown in Fig.3-5. As for case 1 and case 2, it seems that Introducing μ does not help to eradicate the invasive species, so the optimal control μ^* is almost identically 0. In the Fig.6, we can clearly see there is a sharper decay in both f and m under optimal control. The decay rate for case 1 and 3 are almost the same under the optimal control.

For large initial population, case 1 and case 3 do not eradicate the whole population no matter how large μ is, which is shown in Fig.7 and Fig.9. We can also conclude that large μ does not help to eradicate the population, and depending on parameters and initial conditions, the population could blow up when g=1. However, even with large initial population, case 2 can always eradicate the whole population as long as we can provide enough μ, which is shown in Fig.8.

VIII. DISCUSSION AND CONCLUSIONS

The use of Trojan sex chromosomes is an approach for eradicating invasive species that have a XY sex determination system and for which it is feasible to force sex reversal. It was clearly established that extinction is possible in the supermale dynamical system as a function of the rate μ of introduction of supermales (*s*). The TYC system depends upon parameters that can be deduced from observations, including the carrying capacity (*K*), the death coefficients (g_1, g_2, g_3), and the birth coefficients (g_2, g_5, g_6), see (7). Further refinement to these parameters should be made from current field data [62-15].

The existence of a bounded absorbing set indicates that for either eradication or invasion the final state of the population is stable. Via Theorem 6.8 we showed that given the initial population and values for the parameters, it is possible to find an explicit time such that for times greater than this, an attractor is reached, in which the population is confined to finite sized sets in *H*. That is there is a compact subset of the phase space, that attracts all trajectories, in the dynamical system. Furthermore, knowing the analytical form of the bounded absorbing sets, helps guide the exploration of the parameter space. This and related problems are the subject of current investigation [33, 32]. We numerically see that the attractor is a one point attractor, depending on parameter values. Furthermore, what is also tried in the numerical experiments, is that the form of μ is changed from a constant, to being state dependent.

Also we tried numerical experiments, where we use a bad source term such as μr^2. In this case we compare (88) - (91), with *g* as a logistic term, with the same system when μ is replaced with μr^2. Surprisingly, this causes a slower decay in the female species, than if one were to use μ, see Fig2. We also tried simulations with μr^3 and μr^4. What we observe is that as the power on the source term increases, the decay rate of the female species gets *slower*. The point here is one can always stop the influx of the feminized supermale, before the actual blow up, if the bad source actually “sped up" the extinction, but this is not seen to happen. It is however worth investigating other source terms, in this context. The optimal control experiments tell us that in general, whether the initial population of the invasive species is large or small, the classic TYC strategy (case 2) is always effective from the perspective of eradication of the invasive species. The larger the initial population of the invasive species, the larger the introduction of superfemales (*s*) has to be to achieve eradication. Optimal control of the TYC system under various parameter regimes is also an area of current investigations [63] that should be pursued further.

We would like to point out that since $g_i, 1 \le i \le 10$ are all constants in some of our simulations, the conditions for global existence via (20)-(23) are not met. That is there can be various range of large initial data, and parameters, for which we have finite time blow up. The point here is to show that in the good parameter range of data and parameters, the asymptotic behavior of the system is a one point attractor, that is, for a given set of parameters and data where we have global existence, the solutions in long time tend towards a steady state (not necessarily spatially uniform in all components). Furthermore, this can be the extinction state, depending on the size parameter μ. Thus the system can always be driven to the extinction state, via the introduced genetically modified organism. This validates our control strategy, and asserts that in principle, we can always combat invasive species even when used in the context of bio-terrorism, via our proposed strategy [53, 54 and 55].

The analysis of global attractors can be helpful to estimate times to extinction in complex spatial domains. We have determined that for Dirichlet boundary conditions on a connected domain there exists an extinction state as a result of the introduction of *s*. However, more complicated geometries or boundary conditions could have an influence in coexistence or extinction. Increasing the level of sophistication of the eradication strategy, the distribution of *s* individuals could be variable in space as opposed to the constant level that has been studied, i.e. *s* could be a population density dependent function intended to minimize the introduction of *s* individuals and therefore minimize costs of implementation. Also under our proposed strategy, the system can always be driven to the extinction state, at an exponential rate. This is seen numerically as well in Fig1, where we compare the trajectories of normal males and females to the function $e^{-(0.012)t}$. Clearly, exponential attraction to the extinction state is seen. It would be interesting to try and rigorously prove the existence of an exponential attractor, in this setting.

The viability of YY individuals remains an open question. The supermale model assumes that phenotypes are stable after maturation, but this could be problematic for species whose sex determination involves many genes, or when there is environmental pressure to feminization or masculinization. To incorporate this we choose death coefficients so that the supermales die at a faster rate than the normal males, as these are not considered as fit as their normal counterparts, [34], and so fitness penalty should be exercised. Another potential problem is hybridization with compatible species, which would extend the eradication pressure beyond the initial target; however, this effect should disappear by the interruption of the influx of *s*. Also it would make for very interesting future work if we could perhaps place sufficient restrictions on the reaction terms in question to show via Simon-Lojasiewicz gradient inequality techniques [49], to show that convergence to the spatially homogenous equilibrium state is guaranteed.

To summarize, we have rigorously shown that introduction of phenotypically manipulated supermales into an established population can lead to local extinction. Moreover, this can be done even if the population dynamics of the species involved, is *not* governed by a logistic type control term.

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