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Simultaneous Effects of Primary and Secondary Toxicants on the Existence of Two competing Populations in an Aquatic Ecosystem: A Mathematical Analysis

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Abstract:- In this paper, the simultaneous effects of primary and secondary toxicants on the existence of two competing populations in an aquatic ecosystem has been studied and analysed using mathematical techniques and tools. A mathematical model is proposed to study the effect of primary toxicant and as well as secondary toxicants which is formed as a result of the presence of a chemical compound in the water of the aquatic body on the survival or extinction of the two competing populations. The model has been formulated using a system of non-linear differential equations. In this model, a separate differential equation has been considered for the formation of secondary toxicant as a result of the reaction of the primary toxicant with the chemical present in the water. The logistic growth population models for the competing species is considered and it has been assumed that the primary toxicant reduces the carrying capacity of the population and secondary toxicant reduces the specific growth rates of both the populations. The mathematical model proposed in this chapter has been analysed using stability theory.

Keyword:- *Toxicants, Logistic Growth Population Models, Competing Species, Stability Theory.*

I. INTRODUCTION

The study of consequences of pollutants/ toxicants on the ecological communities is of great attention to preserve and conserve the biological species in a given ecosystem. It is known that no population in nature live in isolation therefore it is important to study the effect of toxicants on the existence of two or more competing species systems. There have been numerous analyses associated to the significances of a single toxicant or two toxicants in the amalgamation including interactive effects of biological species in aquatic environment.

In recent decades, the impact of a single toxicant on various populations has been studied using mathematical model (Wallis 1975; Hallam and Clark 1982; Hallam et al 1983a, b; Hallam and De luna 1984; De Luna and Hallam 1987; Barber et al 1988; Freedman and Shukla 1991; Misra and saxena 1991; Misra et al 2000; Misra, Meitei and Rathore 2002; Misra and Meitei 2004; Misra, Jadon and Meitei 2005). In particular Hallam et al (1983) studied the effect of toxicant emitted into the environment on a population by assuming that the growth rate of the population density depends upon the uptake concentration of the toxicant by this population but did not considered the effect of the environmental toxicant on the carrying capacity. Further, Shukla and Dubey (1996) proposed and analysed a non-linear model to study the simultaneous effect of two toxicants, one being more toxic than the other, on a biological population using the stability theory of differential equations.

Acid lowers the pH levels in water bodies below what is required for survival of aquatic life and increases the toxicity of metals (Anita Chaturvedi, Kokila Ramesh, Vatsala, 2017). Aquatic environment is getting polluted by many different types of toxic metal which are discharged from the industries and agricultural fields(Asha Bharathi A, Anita Chaturvedi , Radha Gupta and Kokila Ramesh, 2015), the toxic waste in a river and its remediation by freshening present a simple mathematical model for river pollution and examine the effect of freshening on the humiliation of toxin(D.V. Ramalinga Reddy A.Sreenivasa Chari ,2013), The effect of a toxicant emitted into the environment from external sources on two competing biological species is proposed and analyzed(J. B. Shukla, A. K. Agrawal, B. Dubey and P. Sinha, 2001; A. kumar, A. K. Agrawal, A. Hasan, A. K. Misra, 2016).

A literature review is both a summary and explanation of the complete and current state of knowledge on a limited topic as found in academic books and journal articles.

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II. MATERIALS & METHODS

In the past several decades, mathematical models have become important tools for analysing and predicting the behaviour of ecological systems but modelling the effects of toxicants on biological populations in both aquatic and terrestrial environment is relatively new area of research in ecotoxicology. It may be pointed out further that most of these studies in the past were experimental and few efforts to understand these phenomena using mathematical models have been made.

In view of the above, therefore, in this chapter, the simultaneous effects of primary and secondary toxicants on the existence of two competing populations in an aquatic ecosystem has been studied and analysed using mathematical model. A mathematical model is proposed to study the effect of primary and secondary toxicants which is formed as a result of the presence of a chemical compound in the water of the aquatic body on the survival or extinction of the two competing populations. The logistic growth population models for the competing species is considered and it has been assumed that the primary toxicant reduces the carrying capacity of the population and secondary toxicant reduces the specific growth rates of both the populations. The mathematical model proposed in this paper has been analysed using stability theory.

A. *Mathematical Model:*

The mathematical model to study the Simultaneous Effects of Primary and Secondary Toxicants on the Existence of Two Competing Species System in the Aquatic Environment is given by the following system.

$$\frac{dP}{dt} = r_0 P - \frac{r_0 P^2}{k(x_1)} - a_3 PH - \beta_1 x_2 P \tag{1.2.1}$$

$$\frac{dH}{dt} = b_0 H - \frac{b_o H^2}{K_1(x_1)} - a_4 PH - m_1 x_2 H \tag{1.2.2}$$

$$\frac{dx_1}{dt} = I_0 - d_2 x_1 - a_2 C x_1 \tag{1.2.3}$$

$$\frac{dx_2}{dt} = b_1 a_2 x_1 C - d_1 x_2 - \alpha_1 P x_2 - m_3 H x_2 \tag{1.2.4}$$

$$\frac{dC}{dt} = Q_0 - d_4 C - a_2 x_1 C \tag{1.2.5}$$

Where

P *and* H=Density of two types of Zooplankton,

x_1=Concentration of primary toxicant in the environment of the population,

x_2=Concentration of secondary toxicant in the environment,

C=Concentration of chemical compound present in the aquatic environment with which x_1 reacts to formx_2 ,

d_1 , d_2 and d_4= Decay rates or depletion rates,
$a_3 and a_4$= Competition rate,

a_2= reaction rate of primary toxicant and chemical compound which is responsible for the formation of secondary,

$\alpha_1, \beta_1, m_1, m_3$= Uptake coefficients of toxicants,
r_0, b_0=Intrinsic growth rates

b_1 = Formation rate of secondary toxicant,
I_0=Input rate of primary toxicant,

Alsod_1 , d_2, d_4 , a_2, a_3 , a_4, , $\alpha_1, \beta_1, m_1, m_3, b_1, b_3$ andI_0 all are positive constants.

B. *Uniform Equilibrium Points:*

The uniform equilibrium points of the model given by (1.2.1)-(1,2,5) are obtained as follows:

The **first equilibrium point** is $E_1(\bar{P}, \bar{H}, \bar{x}_1, \bar{x}_2, \bar{C})$ in which, $\bar{P} = 0$, $\bar{H} = 0$,

$$\bar{x}_1 = \frac{I_0}{d_2 + a_2\bar{C}} \;, \quad \bar{x}_2 = \frac{b_1 \bar{x}_1 a_2 \bar{C}}{d_1}$$

And $\bar{C} = \frac{-A+\sqrt{A^2+4d_4 d_2 a_2 Q_0}}{2d_4 a_2} > 0$

If $d_4 d_2 + a_2 I_0 > a_2 Q_0$

where $A = d_4 d_2 + a_2 I_0 - a_2 Q_0$

The **second equilibrium point** is $E_2(\bar{\bar{P}}, \bar{\bar{H}}, \bar{\bar{x}}_1, \bar{\bar{x}}_2, \bar{\bar{C}})$ in which

$$\bar{\bar{P}} = 0, \quad \bar{\bar{C}} = \bar{C}, \quad \bar{\bar{x}}_1 = \frac{I_0}{d_2 + a_2\bar{\bar{C}}},$$

$$\bar{\bar{x}}_2 = \frac{b_1 \bar{\bar{x}}_1 \bar{\bar{C}} a_2}{d_1 + m_3 \bar{\bar{H}}}, \quad \bar{\bar{H}} = \frac{k_1(\bar{\bar{x}}_1)}{b_0}(b_0 - m_1 \bar{\bar{x}}_2) > 0$$

The **third equilibrium point** is $E_3\left(\widehat{\bar{P}}, \widehat{\bar{H}}, \widehat{\bar{x}}_1, \widehat{\bar{x}}_2, \widehat{\bar{C}}\right)$ in which

$$\widehat{\bar{H}} = 0 \;, \quad \widehat{\bar{C}} = \bar{C}, \quad \widehat{\bar{x}}_1 = \frac{I_0}{d_2 + a_2\widehat{\bar{C}}} \;,$$

$$\widehat{\bar{x}}_2 = \frac{b_1 \widehat{\bar{x}}_1 \widehat{\bar{C}} a_2}{d_1 + \alpha_1 \widehat{\bar{P}}} \;, \quad \widehat{\bar{P}} = \frac{k(\widehat{\bar{x}}_1)}{r_0}(r_0 - \beta_1 \widehat{\bar{x}}_2)$$

The **Fourth equilibrium point** is $E_2(\tilde{P}, \tilde{H}, \tilde{x}_1, \tilde{x}_2, \tilde{C})$ in which

$$\tilde{C} = \bar{C} \;, \quad \tilde{x}_1 = \bar{x}_1 \;, \quad \tilde{x}_2 = \frac{b_1 \tilde{x}_1 a_2 \tilde{C}}{d_1 + \alpha_1 \tilde{P} + m_3 \tilde{H}} \;,$$

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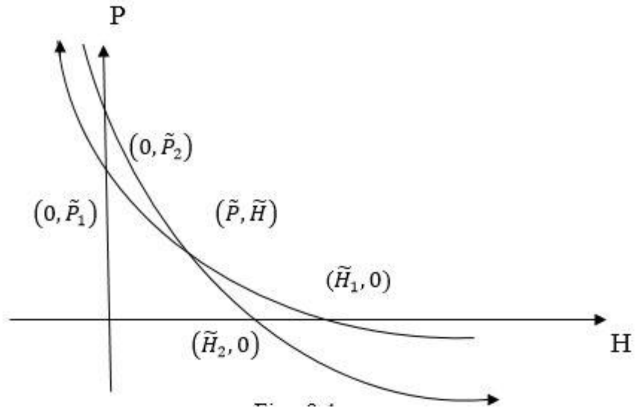
For the existence of $(\tilde{P},\tilde{H})$ the two isoclines $f_1(\tilde{P},\tilde{H})$ and $f_2(\tilde{P},\tilde{H})$ are given by-

$$f_1(\tilde{P},\tilde{H}) = \{r_0 k(\tilde{x}_1) - r_0\tilde{P} - a_3\tilde{H}k(\tilde{x}_1)\}(d_1 + \alpha_1\tilde{P} + m_3\tilde{H}) - \beta_1 k(\tilde{x}_1)A_1 = 0,$$

$$f_2(\tilde{P},\tilde{H}) = \{b_0 k_1(\tilde{x}_1) - b_0\tilde{H} - a_4\tilde{P}k_1(\tilde{x}_1)\}(d_1 + \alpha_1\tilde{P} + m_3\tilde{H}) - m_1 k_1(\tilde{x}_1)A_1 = 0$$

Where, $A_1 = b_1\tilde{x}_1 a_2\tilde{C}$, $f_1(0,\tilde{H}) = 0$ gives one positive root $\tilde{H}_1$ provided $a_3 d_1 > m_3 r_0$ and $r_0 d_1 > \beta_1 A_1$
$f_1(\tilde{P},0) = 0$ gives one positive root $\tilde{P}_1$ provided $d_1 > \alpha_1 k(\tilde{x}_1)$ and $r_0 d_1 > m_3 A_1$
$f_2(0,\tilde{H}) = 0$ gives one positive root $\tilde{H}_2$ provided $d_1 > m_3 k_1(\tilde{x}_1)$ and $b_0 d_1 > m_1 A_1$
$f_2(\tilde{P},0) = 0$ gives one positive root $\tilde{P}_2$ provided $a_4 d_1 > b_0\alpha_1$ and $b_0 d_1 > m_1 A_1$

From the figure shown here it is clear that the two isoclines intersect at $(\tilde{P},\tilde{H})$ under the following conditions $\tilde{P}_2 > \tilde{P}_1$ and $\tilde{H}_1 > \tilde{H}_2$. (1.3A)



The two isoclines intersect with the condition (1.3A) and the two lines exist if

$$\frac{dP}{dH} < 0 \; for f_1(\tilde{P},\tilde{H}) and f_2(\tilde{P},\tilde{H})$$

Therefore, we have

$$\frac{dP}{dH} < \frac{a_3 k(\tilde{x}_1)\varphi_1 - m_3\theta_1}{\alpha_1\theta_1 - r_0\varphi_1} < 0 \text{ for}$$

$$f_1(\tilde{P},\tilde{H}) = 0$$

and

$$\frac{dP}{dH} < \frac{b_0\varphi_2 - m_3\theta_2}{\alpha_1\theta_2 - a_4 k_1(\tilde{x}_1)\varphi_2} < 0 \text{ for } f_2(\tilde{P},\tilde{H}) = 0$$

where,

$\varphi_1 = (d_1 + \alpha_1\tilde{P} + m_3\tilde{H})$,
$\theta_1 = r_0 k(\tilde{x}_1) - r_0\tilde{P} - a_3\tilde{H}k(\tilde{x}_1)$ and
$\varphi_2 = (d_1 + \alpha_1\tilde{P} + m_3\tilde{H})$
$\theta_2 = b_0 k_1(\tilde{x}_1) - b_0\tilde{H} - a_4\tilde{P}k_1(\tilde{x}_1)$

III. RESULT & DISCUSION

A. Linear Stability Analysis:

The local stability analysis of the equilibrium points can be studied from the variational matrix of the mathematical model given by (1.2.1) to (1.2.5) as follows. The variational matrix about the equilibrium point E_1 is given by the following matrix V_1^T:

$$V_1^T = \begin{bmatrix} -(\beta_1\bar{x}_2 - r_0) & 0 & 0 & 0 & 0 \\ 0 & -(m_1\bar{x}_2 - b_0) & 0 & 0 & 0 \\ 0 & 0 & -(d_2 + a_2\bar{C}) & 0 & -a_2\bar{x}_1 \\ -\alpha_1\bar{x}_2 & -m_3\bar{x}_2 & b_1 a_2\bar{C} & -d_1 & b_1 a_2\bar{x}_1 \\ 0 & 0 & -a_2\bar{C} & 0 & -(a_2\bar{x}_1 + d_4) \end{bmatrix}$$

(1.3A)

The characteristics equation of V_1^T is given as follows:

$$\left|V_1^T - \lambda I\right| = 0$$

After the simplification (by the software Mathemtica-5) we find that values of λ are negative if the following conditions are satisfied.

$\beta_1\bar{x}_2 > r_0$ and $m_1\bar{x}_2 > b_0$ (1.4A)

If the above conditions are not satisfied than the equilibrium point E_1 is unstable.

Therefore the equilibrium point E_1 is locally unstable .Similarly, on solving the characteristics equation of V_2^T and V_3^T we find that the equilibrium points E_2 and E_3 are stable if the respective conditions as given below are satisfied.

$\beta_1\bar{\bar{x}}_2 + a_3\bar{\bar{H}} > r_0$ and $m_1\tilde{\tilde{x}}_2 + a_4\tilde{P} > b_0$ (1.4B)

For the linear stability analysis of the equilibrium point E_4, first linearize the system (1.2.1) to (1.2.5) about E_2 using the following linear transformations:

$P = \tilde{P} + n_1, H = \tilde{H} + n_2, x_1 = \tilde{x}_1 + n_3,$
$x_2 = \tilde{x}_2 + n_4, C = \tilde{C} + n_5.$

And then neglect the higher powers and the products of the perturbations to obtain the following linear system –

$$\frac{dn_1}{dt} = -\frac{r_0\tilde{P}}{k(\tilde{x}_1)}n_1 - a_3\tilde{P}n_2 - \frac{r_0\tilde{P}^2 k_2}{k^2(\tilde{x}_1)}n_3 - \beta_1\tilde{P}n_4 \quad (1.4.1)$$

$$\frac{dn_2}{dt} = -a_4\tilde{P}n_1 - \frac{b_0\tilde{H}}{k_1(\tilde{x}_1)}n_2 - \frac{b_0\tilde{H}^2 k_{21}}{k_1{}^2(\tilde{x}_1)}n_3 - m_1\tilde{H}n_4 \quad (1.4.2)$$

$$\frac{dn_3}{dt} = -(d_2 + a_2\tilde{C})n_3 - a_2\tilde{x}_1 n_5 \quad (1.4.3)$$

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$\frac{dn_4}{dt} = -\alpha_1 \tilde{x}_2 n_1 - m_3 \tilde{x}_2 n_2 + b_1 a_2 \tilde{C} n_3 - (d_1 + \alpha_1 \tilde{P} + m_3 \tilde{H}) n_4 + b_1 a_2 \tilde{x}_1 n_5$ (1.4.4)

$\frac{dn_5}{dt} = -a_2 \tilde{C} n_3 - (d_4 + a_2 \tilde{x}_1) n_5$ (1.4.5)

Consider the Liapunov function $\boldsymbol{X}$ as:

$$X = \frac{1}{2}(n_1^2 + A_1 n_2^2 + A_2 n_3^2 + A_3 n_4^2 + A_4 n_5^2),$$

$where, A_i > 0 \ \forall i = 1,2,3,4$, are arbitrary positive constants.

Differentiating $\boldsymbol{X}$ w.r.t. $\boldsymbol{t}$ r and using (1.4.1) to (1.4.5) in $\frac{dX}{dt}$, we get :

$$\frac{dX}{dt} = n_1\left\{-\frac{r_0\tilde{P}}{k(\tilde{x}_1)}n_1 - a_3\tilde{P}n_2 - \frac{r_0\tilde{P}^2k_2}{k^2(\tilde{x}_1)}n_3 - \beta_1\tilde{P}n_4\right\} + A_1n_2\left\{-a_4\tilde{P}n_1 - \frac{b_0\tilde{H}}{k_1(\tilde{x}_1)}n_2 - \frac{b_0\tilde{H}^2k_{21}}{k_1{}^2(\tilde{x}_1)}n_3 - m_1\tilde{H}n_4\right\} + A_2n_3\{-(d_2 + a_2\tilde{C})n_3 - a_2\tilde{x}_1n_5\}$$
$$+ A_3n_4\{-\alpha_1\tilde{x}_2n_1 - m_3\tilde{x}_2n_2 + b_1a_2\tilde{C}n_3 - (d_1 + \alpha_1\tilde{P} + m_3\tilde{H})n_4 + b_1a_2\tilde{x}_1n_5\} + A_4n_5\{-a_2\tilde{C}n_3 - (d_4 + a_2\tilde{x}_1)n_5\}$$

Now using the inequality $a^2 + b^2 \geq \pm 2ab$ in the R.H.S. of $\frac{dX}{dt}$, we obtain

$\frac{dX}{dt} \leq -(S_1 n_1^2 + S_2 n_2^2 + S_3 n_3^2 + S_4 n_4^2 + S_5 n_5^2)$ (1.4.6)

Here $\frac{dX}{dt}$ is negative definite only when, $S_i > 0, \forall i = 1,2,3,4,5.$

Thus we find that the fourth equilibrium point E_4 is locally asymptotically stable under the conditions given by:

$S_i > 0$, $\forall = 1,2,3,4,5$ where,

$$S_1 = \frac{r_0\tilde{P}}{k(\tilde{x}_1)} - \frac{1}{2}\left(a_3\tilde{P} + \frac{r_0\tilde{P}^2k_2}{k^2(\tilde{x}_1)} + \beta_1\tilde{P} + a_4\tilde{P}A_1 + \alpha_1\tilde{x}_2\right)$$

$$S_2 = \frac{b_0\tilde{H}}{k_1(\tilde{x}_1)} - \frac{1}{2}\left(a_3\tilde{P} + \frac{b_0\tilde{H}^2k_{21}}{k_1{}^2(\tilde{x}_1)}A_1 + \beta_1\tilde{P} + a_4\tilde{P}A_1 + m_1\tilde{H}A_1 + A_3m_3\tilde{x}_2\right)$$

$$S_3 = A_2(d_2 + a_2\tilde{C}) - \frac{1}{2}\left(\frac{r_0\tilde{P}^2k_2}{k^2(\tilde{x}_1)} + + \frac{b_0\tilde{H}^2k_{21}}{k_1{}^2(\tilde{x}_1)}A_1 + A_2a_2\tilde{x}_1 + A_3b_1a_2\tilde{C} + A_4a_2\tilde{C}\right)$$

$$S_4 = A_3(d_1 + \alpha_1\tilde{P} + m_3\tilde{H}) - \frac{1}{2}(A_3\alpha_1\tilde{x}_2 + + \beta_1\tilde{P} + m_1\tilde{H}A_1 + A_3m_3\tilde{x}_2 + A_3b_1a_2\tilde{C} + A_4a_2\tilde{C})$$

B. Non-linear Stability Analysis:

For non-linear stability analysis of equilibrium point E_4 assume the region Δ is bounded and given by:

$$\Delta = \left\{\begin{matrix}(\tilde{P},\tilde{H},\tilde{x}_1,\tilde{x}_2,\tilde{C}): 0 < \tilde{P}^l \leq \tilde{P} \leq \tilde{P}^u, 0 < \tilde{H}^l \leq \tilde{H} \leq \tilde{H}^u, 0 < \tilde{x}_1^l \leq \tilde{x}_1 \leq \tilde{x}_1^u, \\ 0 < \tilde{x}_2^l \leq \tilde{x}_2 \leq \tilde{x}_2^u, 0 < \tilde{C}^l \leq \tilde{C} \leq \tilde{C}^u.\end{matrix}\right\}$$

If the following inequalities hold, then the equilibrium point E_4 is nonlinearly stable.

$$\frac{r_0}{k(\tilde{x}_1)} > \frac{1}{2}\left(a_3 + \beta_1 + a_4 + \alpha_1\tilde{x}_2 + \frac{r_0P^lk_2}{k^2(\tilde{x}_1)}\right)$$

$$\frac{b_0}{k_1(\tilde{x}_1)} > \frac{1}{2}\left(a_3 + m_1 + a_4 + \frac{b_0Hk_{21}}{k_1{}^2(\tilde{x}_1)} + m_3\tilde{x}_2\right)$$

$$(d_2 + a_2\tilde{C}) > \frac{1}{2}\left(\frac{r_0P^lk_2}{k^2(\tilde{x}_1)} + a_2x_1^l + b_1a_2\tilde{C} + a_2\tilde{C}\right)$$

$$(d_1 + \alpha_1P^l + m_3H^l) > \frac{1}{2}(\beta_1 + m_1 + \alpha_1\tilde{x}_2 + m_3\tilde{x}_2 + b_1a_2\tilde{C} + b_1a_2\tilde{x}_1)$$

$$\left(d_4 + \frac{a_2x_1^l}{2}\right) > \frac{1}{2}(b_1a_2\tilde{x}_1 + a_2\tilde{C})$$

Proof of Theorem 1.5.1:

Using the transformations

$$P = \tilde{P} + n_1, H = \tilde{H} + n_2, x_1 = \tilde{x}_1 + n_3, x_2 = \tilde{x}_2 + n_4, C = \tilde{C} + n_5.$$

The system (1.2.1) to (1.2.5) reduces to:

$\frac{1}{\tilde{P}+n_1}\frac{dn_1}{dt} = -\frac{r_0}{k(\tilde{x}_1)}n_1 - a_3n_2 - \frac{r_0Pk_2}{k^2(\tilde{x}_1)}n_3 - \beta_1n_4$ (1.5.1)

$\frac{1}{\tilde{H}+n_2}\frac{dn_2}{dt} = -a_4n_1 - \frac{b_0}{k_1(\tilde{x}_1)}n_2 - \frac{b_0Hk_{21}}{k_1{}^2(\tilde{x}_1)}n_3 - m_1n_4$ (1.5.2)

$\frac{dn_3}{dt} = -(d_2 + a_2\tilde{C})n_3 - a_2x_1n_5$ (1.5.3)

$\frac{dn_4}{dt} = \alpha_1\tilde{x}_2n_1 - m_3\tilde{x}_2n_2 + b_1a_2\tilde{C}n_3 - (d_1 + \alpha_1P + m_3H)n_4 + b_1a_2\tilde{x}_1n_5$ (1.5.4)

$\frac{dn_5}{dt} = -a_2\tilde{C}n_3 - (d_4 + a_2\tilde{x}_1)n_5$ (1.5.5)

Consider the positive definite function:

$$Y = \left\{n_1 - \tilde{P}log\left(1 + \frac{n_1}{\tilde{P}}\right)\right\} + \left\{n_2 - \tilde{H}log\left(1 + \frac{n_2}{\tilde{H}}\right)\right\} + \frac{1}{2}(n_3^2 + n_4^2 + n_5^2)$$

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Differentiating ***Y*** w.r.t.***t***, using (1.5.1) to (1.5.5) and the inequality $a^2 + b^2 \geq \pm 2ab$ in $\frac{dY}{dt}$, we have

$$\begin{aligned}\frac{dY}{dt} \leq -\Bigg[&\left\{\frac{r_0}{k(\tilde{x}_1)} - \frac{1}{2}\left(a_3 + \beta_1 + a_4 + \alpha_1\tilde{x}_2 + \frac{r_0 P^l k_2}{k^2(\tilde{x}_1)}\right)\right\} n_1^2 \\ &+ \left\{\frac{b_0}{k_1(\tilde{x}_1)} - \frac{1}{2}\left(a_3 + m_1 + a_4 + \frac{b_0 H k_{21}}{k_1^{\,2}(\tilde{x}_1)} + m_3\tilde{x}_2\right)\right\} n_2^2 \\ &+ \left\{(d_2 + a_2\tilde{C}) - \frac{1}{2}\left(\frac{r_0 P^l k_2}{k^2(\tilde{x}_1)} + a_2 x_1^l + b_1 a_2 \tilde{C} + a_2\tilde{C}\right)\right\} n_3^2 \\ &+ \left\{(d_1 + \alpha_1 P^l + m_3 H^l) - \frac{1}{2}(\beta_1 + m_1 + \alpha_1\tilde{x}_2 + m_3\tilde{x}_2 + b_1 a_2\tilde{C} + b_1 a_2 \tilde{x}_1)\right\} n_4^2 \\ &+ \left\{\left(d_4 + \frac{a_2 x_1^l}{2}\right) - \frac{1}{2}(b_1 a_2 \tilde{x}_1 + a_2\tilde{C})\right\} n_5^2\Bigg]\end{aligned}$$

Now we find that $\frac{dY}{dt}$ is negative definite if the following conditions hold good:

$$\frac{r_0}{k(\tilde{x}_1)} > \frac{1}{2}\left(a_3 + \beta_1 + a_4 + \alpha_1\tilde{x}_2 + \frac{r_0 P^l k_2}{k^2(\tilde{x}_1)}\right) \tag{1.5.6}$$

$$\frac{b_0}{k_1(\tilde{x}_1)} > \frac{1}{2}\left(a_3 + m_1 + a_4 + \frac{b_0 H k_{21}}{k_1^{\,2}(\tilde{x}_1)} + m_3\tilde{x}_2\right) \tag{1.5.7}$$

$$(d_2 + a_2\tilde{C}) > \frac{1}{2}\left(\frac{r_0 P^l k_2}{k^2(\tilde{x}_1)} + a_2 x_1^l + b_1 a_2\tilde{C} + a_2\tilde{C}\right) \tag{1.5.8}$$

$$(d_1 + \alpha_1 P^l + m_3 H^l) > \frac{1}{2}(\beta_1 + m_1 + \alpha_1\tilde{x}_2 + m_3\tilde{x}_2 + b_1 a_2\tilde{C} + b_1 a_2\tilde{x}_1) \tag{1.5.9}$$

$$\left(d_4 + \frac{a_2 x_1^l}{2}\right) > \frac{1}{2}(b_1 a_2\tilde{x}_1 + a_2\tilde{C}) \qquad (1.5.10)$$

(1.5.10)

Hence, E_4 is nonlinearly asymptotically stable in the region Δ with the conditions (1.5.6) to (1.5.10).If the above conditions are not satisfied than the equilibrium point is globally unstable

Study results normally refer to direct answers to your research questions that you generate from the data. Discussion is about interpreting your study results.

IV. CONCLUSION

The stability analysis of the four feasible equilibrium points E_1, E_2, E_3 and E_4 shows that the first equilibrium points E_1 is stable with the conditions involving system parameters but the conditions for the stability of E_1 look less feasible as compare the stability conditions for the equilibrium points E_2 and E_3 and also non-trivial equilibrium point E_4 is linearly asymptotically stable (conditionally). From the stability analysis of the first equilibrium point E_1 it may be suggested that two types of competing species would go to elimination due to toxicants-pollutants present in the water and the stability analysis of the equilibrium point $E_2\, and\, E_3$, implies that out of the two types of zooplanktons one will go to extinction, the other will survive. From the stability analysis of the non-trivial positive equilibrium point, it is also observed that both the two types of competing zooplanktons coexist at lower equilibrium values due to the effect of toxicants and chemicals.

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