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Review on Mathematical Modelling in Water Pollution

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Abstract:- This paper is a review on modelling the effects of water pollution and eutrophication on the survival or extinction of aquatic species. The models are formulated using the system of non-linear ordinary differential equations. The systems are modelled in consideration of the variables like concentration of dissolved oxygen(DO), concentration of acid and metal, density of resources(phytoplankton), nutrient concentration and density of fish population. Using stability analysis and numerical simulation the conditions for the existence of equilibrium points have been drawn. And the criteria for the survival or extinction of the aquatic species have been obtain.

***Keyword*:-** *Mathematical Modelling, Eutrophication, Dissolved Oxygen(DO), Nutrients.*

I. INTRODUCTION

A community of plants, animals and other organisms along with their environment including the air, water and soil is an ecosystem. Everything in an ecosystem is linked to each other. Acid precipitation and eutrophication have become an important issue and research area in the present scenario. The ecological effects of acid rain are most clearly seen in aquatic environment, such as lakes, streams and marshes. A great quantity of the toxicants and contaminants enter into the ecosystem continuously which threaten the survival of the exposed population including human beings. Aquatic environment is getting polluted by many different types of toxic metals which are discharged from the industries and agricultural fields. The chief pollutants which are produced by the industries are heavy metals and radioactive substances. In the agricultural fields fertilizers, pesticides and insecticides are used for the production and control of plant diseases. The fertilizers, pesticides and insecticides containing toxic metals reach the water bodies though the runoff, causing harms to aquatic life. The acid rain changes the pH value of water which is harmful to the aquatic population. it has been found that the toxicity of metals in aquatic environment increases due to the acidity of water. For example, many fish species die when the pH is less than five. In addition to the direct effect to direct effects of acid on fish, the acid mobilizes such as aluminium from the surrounding soil from which it enters lakes by runoff. When combined with high acidity, aluminium is toxic to aquatic species.

Eutrophication is when a body of water becomes overly enriched with minerals and nutrients that induce excessive growth of plants and algae. This process may result in oxygen depletion of the water body and thereby increasing the concentration of phytoplankton and algae. Due to oxygen deficit and loss of transparency, growth of much aquatic population such as fish is adversely affected.
Considering these aspects therefore, mathematical models have been proposed to study the effects of eutrophication and the combined effects of acidity and toxicity of metals on the growth and existence of aquatic populations incorporating nutrient effects on the system. The systems are modelled by considering the parameters like concentration of dissolved oxygen(DO), concentration of acid and metal, density of resources (phytoplankton), nutrient concentration and density of fish population.

II. MATHEMATICAL MODELS

A. *Mathematical Modelling & analysis of the depletion of dissolved oxygen in eutrophied water bodies affected by organic pollutants. [J.B. Shukla, A.K. Misra, Peeyush Chandra, (2007, May 31)]*

In this paper a mathematical modelling is proposed to study the simultaneous effect of water pollution and eutrophication on the concentration of dissolved oxygen.

Model:

$$\frac{dT}{dt} = Q - \alpha_0 T - \frac{K_1 TB}{K_{12} + K_{11} T},$$

$$\frac{dB}{dt} = \frac{\lambda_1 K_1 TB}{K_{12} + K_{11} T} - \alpha_1 B - \lambda_{10} B^2,$$

$$\frac{dn}{dt} = q + \pi\delta S - \alpha_2 n - \frac{\beta_1 na}{\beta_{12} + \beta_{11} n},$$

$$\frac{da}{dt} = \frac{\theta_1 \beta_1 na}{\beta_{12} + \beta_{11} n} - \alpha_3 a - \beta_{10} a^{10}$$

$$\frac{dS}{dt} = \pi_1 \alpha_1 B + \pi_2 \alpha_3 a - \delta S,$$

$$\frac{dC}{dt} = q_c - \alpha_4 C - \lambda_{12} \frac{K_1 TB}{K_{12} + K_{11} T} - \lambda_{11} \alpha_1 B + \lambda_{22} a - \delta_1 S,$$

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Where T(0)≥ 0, B(0)≥ 0, n(0)≥ 0, a(0)≥ 0, S(0)≥ 0, C(0)≥ 0.

Here $\alpha_2, \alpha_0, and\ \alpha_4$ are natural depletion rate coefficient of nutrients, organic pollutants and dissolved oxygen, which are positive and constants. The coefficient α_1 represents the natural depletion rate as well as predating rate of bacterial population whereas α_3 represents the similar coefficients for algae, both are assumed to be positive constants. The positive constants λ_{10} and β_{10} are coefficients representing crowding (flaking off coefficients) of bacteria and algae, respectively, with respect to the aquatic habitat. Further as π, π_1, π_2 are fractional proportionality constants, we have $0 < \pi, \pi_1, \pi_2 < 1$.

➢ *Conclusion:*

This model has been analysed by using stability analysis of differential equations and numerical simulations by considering its various feasible equilibrium points. It has been shown that the simultaneous effect of water pollution and eutrophication is much more in decreasing the concentration of decreasing oxygen and the survival of aquatic species is much more uncertain.

B. A Mathematical Approach To Study The Effects Of Pollutants/Toxicants in Aquatic Environment. [Anita Chaturvedi, Kokila Ramesh, Vatsala G A, (2017)]

In this paper, a system of non-linear differential equations has been analysed using the parameters acid and metal concentrations, nutrients, favourable resources and fish population. Taking these factors, the author has proposed a mathematical model incorporating crowding effect of fish.

Model:

$$\frac{dS}{dt} = S_0 - aS - gSP - \alpha(T_1 + qC_m)S + KcP + KbF + K_1F^2$$

$$\frac{dP}{dt} = gSP - cP - fFP$$

$$\frac{dF}{dt} = fPF - bF - KF^2$$

$$\frac{d(T_1 + qC_m)}{dt} = Q_0 - \alpha(T_1 + qC_m) - \alpha_1(T_1 + qC_m)S$$

With the initial conditions:

$$S(0) = S_{10} > 0, P(0) => P_{10} > 0, F(0) = F_{10} > 0, T(0) = T_{10} > 0$$

$S_0, a, b, c, f, g, \alpha_1, \alpha, Q_0, K, K_1$, are positive constants.

Where Q_0 = constant input rate of pollutant in water, $S_0 =$ constant nutrient rate the in water, α=Nutrients leaching rate, g= Rate of consumption of nutrients by the resource population, α_1 = Depletion rate of nutrient due to pollution concentration in water, f=Specific rate of predation of fish on resource population, c = Natural death rates of resource, b = Natural death rates of fish populations, α = natural washout rates of acid and metal respectively, K ($0 < K < 1$) = proportionate amount of resource and fish populations that is being recycled back to the nutrient pool after death.

➢ *Conclusion:*

The conditions for equilibrium points are drawn using stability analysis. It has been observed that the interior equilibrium is sensitive to specific rate of predation of fish on resource population. It has also been observed that the variables reach the equilibrium values both using numerical simulation and numerical examples. And the simulated and calculated values match with each other. And the interactions between the variables have been studied in the form of phase plots.

C. Combined Effects of Acids and Metals on the survival of Resource Based Population Incorporating Nutrient Recycling: A Mathematical Model. [Asha Bharathi A.T, Anita Chaturvedi, Radha Gupta and Kokila Ramesh, (2015)]

In this paper, the author has formulated a model using the system of non-linear differential equations. In the model there are five parameters, concentration of acid, concentration of metal, density of phytoplankton, density of fish population and nutrient concentration.

Model:

$$\frac{dT_1}{dt} = H_0 - \alpha_1T_1 - \beta_1T_1S$$

$$\frac{dC_m}{dt} = Q_0 - \alpha_2C_m - \beta_2C_mS$$

$$\frac{dB}{dt} = gSB - cB - fBN$$

$$\frac{dN}{dt} = fBN - bN$$

$$\frac{dS}{dt} = S_0 - aS - gSB - \beta_1T_1S - \beta_2C_mS + KcB + KbN$$

With the initial conditions

$$T_1(0) = T_{10} > 0, C_w(0) = C_{10} > 0, B(0) = B_{10} > 0, N(0) = N_{10} > 0, S(0) = S_{10} > 0,$$

Where T_1 = Concentration of acid in water, C_w = Concentration of metal in water, B= Density of favourable resource, N = Density of fish population, S = Concentration of nutrient, H_0 = Total input rate of acid, Q_0 = Total input rate of metal. α_1 and α_2 = natural washout rates of acid and metal respectively. β_1 β_2 = Depletion rate of nutrient due to acid and metal respectively in water. g= Rate of consumption of nutrients by the resource population. f=Specific rate of predation of fish on resource population. b, c = Natural death rates of resource and fish populations. S_0 = constant nutrient input in the water. a=Nutrients

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leaching rate. K ($0< K < 1$) = proportionate amount of resource and fish populations that is being recycled back to the nutrient pool after death.

$$H_0,\ Q_0, S_{0,} a, b, c, f, g, \alpha_2, \beta_1,\ \beta_2,\ a_1,\ Q_0$$

Are positive constants.

- *Conclusion:*

The conditions for local stability and feasible equilibrium points (non-living, fish extinct and interior) has been determined. It has been observed that interior equilibrium is sensitive to specific rate of predation of fish on resource population. It is found that when the equilibrium level of nutrients increases then the equilibrium of acid and metal goes down. It was also observed that the amount of nutrient is increases if the population densities of resources and fish population goes up. The criteria for the survival or extinction of the species has been obtained using numerical simulation. Stability of the system is explained analytically as well as graphically.

*D. Modelling Effect of Eutrophication on the survival of Fish Population Incorporating Nutrient Recycling. [*Anita Chaturvedi, O.P. Misra, (2010, September 15)*]*

In this paper the author has proposed a non-linear mathematical model to study the survival or extinction of fish population under the adverse effect of eutrophication incorporating the impact of direct and indirect recycling of the nutrients.

Model without Diffusion:

$$\frac{\partial H}{\partial t} = \beta P - \alpha_h H,$$

$$\frac{\partial F}{\partial t} = r(D)F - \frac{r_0 F^2}{k(N)} + D_0 \frac{\partial^2 F}{\partial z^2},$$

$$\frac{\partial C}{\partial t} = -k_1 CP - d_B(N) + k_2(m)$$

$$(C_3 - C) + D_1 \frac{\partial^2 C}{\partial z^2},$$

$$\frac{\partial P}{\partial t} = 1 - rP - \frac{d_1 PN}{b + P} + a_1 m_d D_N + D_2 \frac{\partial^2 P}{\partial z^2},$$

$$\frac{\partial N}{\partial t} = \frac{d_1 PN}{b + P} - aN - gN^2 + D_3 \frac{\partial^2 N}{\partial z^2},$$

$$\frac{\partial D_N}{\partial t} = I_N + \alpha a N - m_d D_N + D_3 \frac{\partial^2 N}{\partial z^2},$$

With the initial conditions which are given as follows:

H(z,0) $= f_1(z) \geq 0, F(z,0) = f_2(z) \geq 0, C(z,0) = f_3(z) \geq 0$

P(z,0) $= f_4(z) \geq 0, H(z) = f_5(z) \geq 0, D_N(z,0) = f_6(z) \geq 0,$

The model is associated with the following boundary conditions z=0, a.

H=H*F=F*, C=C*, P=P*, N=N*,$D_N = D_N$*H*, F*, C*, P*, N*, D_N* are the equilibrium values (or steady states).

For the analysis of the model given above we assume the following forms for the functions r(d), k(N), $d_b(N)$ and $k_2(m)$. D is assumed to be

D=$C_5 - C$.

m(t)=$\frac{m_0}{1+H(t)}$, $k_2(m) = \frac{k_{20} m}{k_{22}+m}$, $d_B(N) = d_{B0} + d_{B1} N$,
$k(N) = k_0 - k_{11} N$, $r(D) = \frac{r_0}{1+D}$

Model without Diffusion:

$$\frac{dH}{dt} = \beta P - \alpha_h H,$$

$$\frac{dF}{dt} = r(D)F - \frac{r_0 F^2}{k(N)},$$

$$\frac{dC}{dt} = -k_1 CP - d_B(N) + k_2(m)(C_3 - C),$$

$$\frac{dP}{dt} = 1 - rP - \frac{d_1 PN}{b + P} + a_1 m_d D_N,$$

$$\frac{dN}{dt} = \frac{d_1 PN}{b + P} - aN - gN^2,$$

$$\frac{dD_N}{dt} = I_N + \alpha a N - m_d D_N,$$

Where, F=fish population, C=concentration of dissolved oxygen, D=oxygen deficit, m=transparency, H=concentration of chlorophyll, N=phytoplankton(Algae and vascular plants), P=concentration of soluble phosphorus, $k_2(m)$ =Reaeration rate, $d_B(N) =$ Removal rate of DO due to respiration by organism, a=Death rate of phytoplankton, r=Death rate of fish, r(D)=Growth rate of fish, K(N)=Carrying capacity of fish, C_5 =Saturated concentration of dissolved oxygen, I=Input rate of P, D_i =Diffusion(dispersal) coefficient where, i=0,1,2,3; D_N =Detritus biomass in terms of the limiting nutrients, I_N =Input of organic nutrients in form of plant detritus, m_d= Plant detritus mineralisation rate, α= Fractions of nutrients released by plants that stays with the system and goes to the herbivore detritus, a=Rate of plant detritus production and $a_h, g, r_0, k_1, a_1, d_1, b, \alpha, \beta$ are positive constants and z is space variables.

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➢ *Conclusion –*

In this paper the model has three feasible steady states E_1, E_2 and E_3. It has been observed that in the presence of diffusion the equilibrium points E_1 and E_2 both are locally stable and the equilibrium point E_3 is linearly as well as non-linearly asymptotically stable. And also, at lower equilibrium level the fish population will exist. And the unstable equilibrium points E_1 and E_2 become stable with diffusion. Also, but in the case of the stability of the equilibrium point E_1 both phytoplankton and population of fishes tend to extinct, but in case of the equilibrium point E_2 only the fish population would tend to extinction.

III. CONCLUSION

In this paper, we have studied effects of eutrophication and water pollution on the concentration of various parameters in a water body such as lake. It has been assumed that the discharge rates of organic pollutants and nutrients by water runoff from agricultural fields are constant. The models have been analysed by using stability theory of differential equations and numerical simulations by considering its various feasible equilibria. It has been shown that the effect of eutrophication and water pollution is more in increasing the concentration of acid, metal and density of resources(phytoplankton). And thereby decreasing the concentration of dissolved oxygen(DO) and nutrient. Thus, it may be postulated that the survival of aquatic species in water body, which is polluted is much more uncertain.

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