Square Difference Labeling of Some Special Graphs

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Abstract:- In this paper, we prove that some special classes of graphs like subdivision graph S(G), $(K_{1,n}^{(1)}, K_{1,n}^{(2)})$, $C_m \Theta \overline{K}_n$, generalized theta graph $\theta[\ell^{[m]}]$, Hanging generalized theta graph, Duplication of all the edges of C_n, Square graph of comb $(P_n \Theta K_1)^2$, $B_{n,n}^2$ are admits square difference labeling (SDL).

***Keywords*:-** *Square Difference Graph (SDG), Subdivision Graph, Crown Graph, Theta Graph, Comb Graph, AMS Subject Classification: 05C78.*

I. INTRODUCTION

The square difference labeling were introduced by Shiama [6]. For our study, we consider a simple, undirected, finite graph and we follow [1, 5] for all terminology and notations.

A dynamic survey on graph labeling is regularly updated by Gallian [2]. K. Manimekalai and K. Thirusangu [4] investigated pair sum labeling of generalized theta graph and subdivision of the edges of the star. E. Esakkiammal proved that the square graph of comb are square difference graph [3].

In section 2, we present some definitions, which are needed for the present work.

In section 3, we prove the existence of square difference for some special graphs.

II. DEFINITIONS

➢ *Definition 2.1*

A function of a graph G = (p, q) is said to be a square difference graph if it admits a bijective function $g : V \rightarrow \{0, 1, 2, \dots p-1\}$ such that the induced function $g^* : E(G) \rightarrow N$ given by $f^*(xy) = |[f(x)]^2 - [f(y)]^2|$ are all distinct, $\forall xy \in E(G)$.[6].

➢ *Definition 2.2*

A subdivision graph S(G) is obtained from G by subdividing each edge of G with a vertex [4].

➢ *Definition 2.3*

Consider G = $(K_{1,n}^{(1)}, K_{1,n}^{(2)})$ is the graph attained by joining apex vertices of stars to a new vertex *x*.

➢ *Definition 2.4*

The corona of two graph G and H is the graph obtained by taking one copy of G and |V(G)| copies of H and attaching each i^{th} apex of G to every vertex in i^{th} copy of H.

➢ *Definition 2.5*

Bistar is the graph attained by joining the apex vertices of two copies of star $K_{1,n}$.

➢ *Definition 2.6*

A generalized theta graph consisting of $k \geq 2$ internal disjoint paths of length *l* with the same end points *u* & *v* and denoted as $\theta[\ell^{[m]}]$ [4].

➢ *Definition 2.7*

Let P_n be a path graph with n vertices. The comb graph is defined as $P_n \Theta K_1$. It has $2n$ vertices and $2n-1$ edges [3].

➢ *Definition 2.8*

The G^2 of an undirected graph G is another graph that has the same set of vertices, but in which two vertices are adjacent when their distance in G is at most 2. [3]

The structure of square graph of comb is given below.

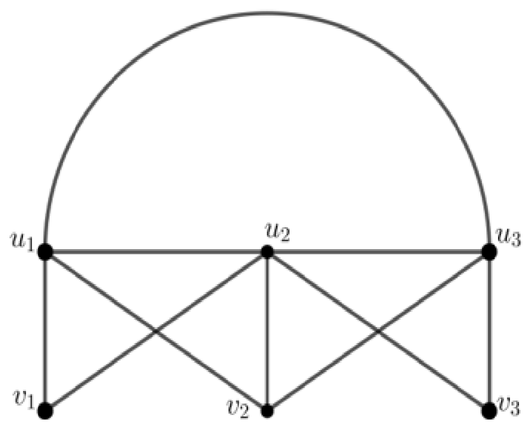

Fig 1:- Square Graph of Comb

III. MAIN RESULTS

A. *Theorem 3.1*

The subdivision of the edges of the star $K_{1,n}$ admits square difference labeling.

➢ *Proof*

Let G be a graph obtained by the subdivision of the edges of the star graph $K_{1,n}$ with $V(G) = \{v, u_i, w_i \ / \ 1 \le i \le n\}$, where u_i & w_i are adjacent vertices of v and u_i respectively and
$E(G) = \{vu_i, u_iw_i \ / \ 1 \le i \le n\}$
Then $|V| = 2n+1$ & $|E| = 2n$
Now define the function $f : V(G) \to \{0, 1, \ldots, 2n\}$ as
$f(v) = 0$
$f(u_i) = 2i-1$
$f(w_i) = 2i$
and the induced function $f^*: E(G) \to N$ given by $f^*(uv) = |[f(u)]^2 - [f(v)]^2|$, $\forall uv \in E(G)$. For the above labeling pattern, we attain the edge labels f^* as,
For $1 \le i \le n$,
$f^*(v, u_i) = (2i-1)^2$
$f^*(u_i, w_i) = 4i-1$

Thus the entire 2n edges receive labels that are all distinct and strictly increasing sequence. Hence the theorem. The example of $S(K_{1,5})$ shown in *figure 2*.

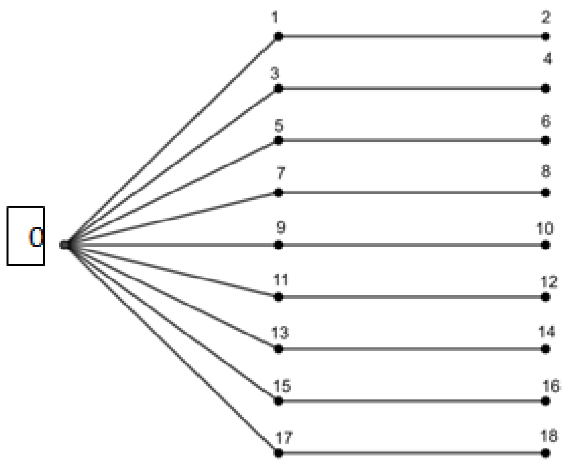

Fig 2:- SDL for $S(K_{1,5})$

B. *Theorem 3.2*

The graph $(K_{1,n}^{(1)}, K_{1,n}^{(2)})$ is a square difference graph.

➢ *Proof*

Consider a graph G with $|V(G)| = 2n+3$. Let x_j and y_j, $j = 1, 2, \ldots$ n are the vertices of $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ respectively and let x and y are the apex vertices of the same, which are adjacent to common vertex w.

Define a mapping $g: V(G) \to \{0, 1, \ldots, 2n+2\}$ as
$g(x) = 0$
$g(y) = 1$
$g(x_j) = j+1$ and
$g(y_j) = f(x_n)+j$, for $1 \le j \le n$
$g(w) = 2n+2$

Then the induced edge function g^* yields the edge labeling as follows.
$g^*(x,x_j) = (j+1)^2$
$g^*(y,y_j) = (n+j+1)^2-1$
$g^*(wx) = (2n+1)^2$
$g^*(wy) = (2n+1)^2-1$

Hence $g^*(e_i) \neq g^*(e_j)$, $\forall ei, ej \in E(G)$. i.e., all the edge labeling are distinct. Therefore the graph G is SD graph. For instance, $(K_{1,5}^{(1)}, K_{1,5}^{(2)})$ are shown below.

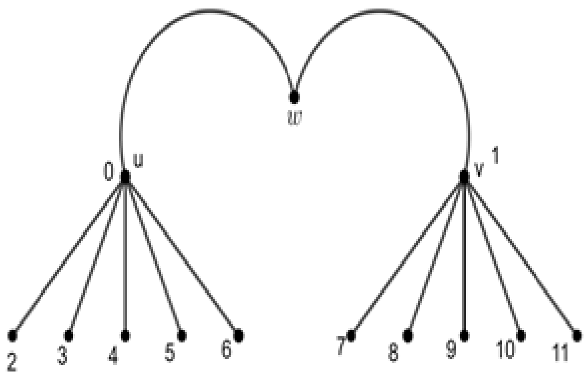

Fig 3:- SDL for $(K_{1,5}^{(1)}, K_{1,5}^{(2)})$

C. *Theorem 3.3*

The generalized crown $C_m \odot \overline{K}_n$ admits SD labeling.

➢ *Proof*

Let the vertex set $V = V_1 \cup V_2$, where
$V_1 = \{v_i \ / \ 1 \le i \le m\}$ and
$V_2 = \{ u_i^{(j)} \ / \ 1 \le i \le m, 1 \le j \le n\}$ and
the edge set $E = E_1 \cup E_2$, where
$E_1 = \{v_1v_2 \ v_3 \ v_4 \ \ldots \ v_m \ v_1\}$ and
$E_2 = \{v_i \, u_i^{(j)} \ / \ 1 \le i \le m, 1 \le j \le n\}$.

Then the cardinality of vertices is mn+4.
Now, define a vertex valued function $f : V \to \{0, 1, \ldots, mn+3\}$ as follows:
For $1 \le i \le m$ and $1 \le j \le n$
$f(v_i) = i-1$
$f(u_i^{(j)}) = m + (i-1)n + j-1$

For the above labeling pattern, the induced function f^* satisfies the condition of square difference labeling and it yields the edge labels as
$f^*(v_i, v_{i+1}) = 2i-1, 1 \le i \le m-1$
$f^*(v_m, v_1) = [f(v_m)]^2$

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$f^*(v_i u_i^{(j)}) = |(i-1)^2-[(j+m-1)+7(i-1)]^2|,\ 1 \le i \le m,\ 1 \le j \le n$.

Thus the labeling of edges of G are distinct and hence the graph $C_m \odot \overline{K}_n$ admits SDL. The graph $C_4 \odot \overline{K}_7$ given below.

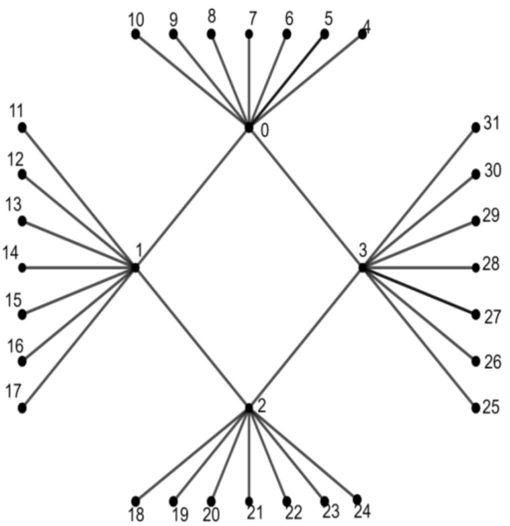

Fig. 4:- SDL for $C_4 \odot \overline{K}_7$

D. Theorem 3.4

The generalized theta graph $\theta[\ell^{[m]}]$ is a SDG.

➢ *Proof*

Consider $G(V, E) = \theta[\ell^{[m]}]$ with $|V(G)| = m(\ell-1)$ and $|E(G)| = \ell m$. Let us consider the following two cases.

- *Case 1:*

***m* is even**

Let $V(G) = V_1 \cup V_2$, where
$V_1 = \{u_{i,j}, v_{i,j} / 1\le i \le \ell-1, 1\le j \le m/2\}$
$V_2 = \{u, v\}$
and $E(G) = E_1 \cup E_2 \cup E_3$, where
$E_1 = \{u_{i,j}\, u_{i+1,j} / 1 \le i \le \ell-2, 1 \le j \le m/2\}$
$E_2 = \{v_{i,j}\, v_{i+1,j} / 1 \le i \le \ell-2, 1 \le j \le m/2\}$
$E_3 = \{uu_{1,j}\ vu_{\ell-1,j},\ uv_{1,j},\ vv_{\ell-1,j}\}$
Define $f : V(G) \to \{0, 1, \ldots, m(\ell-1)+1\}$ as follows:
$f(u) = m(\ell-1)+1$
$f(v) = m(\ell-1)$
$f(u_{i,j}) = 2(i-1)+2(\ell-1)(j-1)$,
$f(v_{i,j}) = 2i-1+2(\ell-1)(j-1)$, for $1 \le i \le \ell-1$, $1 \le j \le m/2$
The induced function $f^* : E \to N$ is defined by $f^*(uv) = |[f(u)]^2- [f(v)]^2|$, $\forall\ uv \in E(G)$, we get the edge labels as follows.
$f^*(e_i) \not\equiv 0 \ (mod\ 8)\ \forall e_i \in E_1$ &
$f^*(e_j) \equiv 0 \ (mod\ 8)\ \forall e_j \in E_2$, here $f^*(e_i) \equiv 0 \ (mod\ 4)$ contained in $f^*(e_j) \equiv 0 \ (mod\ 8)$ i.e., $f^*(u_{i,j}\, u_{i+1,j}) < f^*(v_{i,j}\, v_{i+1,j})$

It is easily observed that $f^*(e_i) \ne f^*(e_j)$, $\forall e_i \ne e_j \in$ E(G).

Similarly, we receive the edge labels for E_3 in the following sub cases:

✓ *Subcase 1.1:*

When ℓ is odd

$f^*(uu_{1,j}) \equiv 1 \ (mod\ 4)$
$f^*(vv_{\ell-1,j}) \equiv 3 \ (mod\ 4)$
$f^*(uv_{1,j}) \equiv 0 \ (mod\ 8) = a$
$f^*(vu_{\ell-1,j}) \equiv 0 \ (mod\ 8) = b,\ (a < b)$

✓ *Subcase 1.2:*

When ℓ is even

$f^*(uu_{1,j}) \equiv 1 \ (mod\ 4)$
$f^*(vv_{\ell-1,j}) \equiv 3 \ (mod\ 8)$
$f^*(uv_{1,j})$ and $f^*(vu_{\ell-1,j})$ are same as sub case 1.1

- *Case 2:*

***m* is odd**

Define $V(G) = V_1 \cup V_2 \cup V_3$ for $1 \le i \le \ell-1$ & $1 \le j \le \lfloor m/2 \rfloor$. Consider V_1 and V_2 are same as defined in *Case 1* and $V_3 = \{w_i\}$

Similarly $E(G) = \bigcup_{k=1}^{4} E_k$, where E_1, E_2, E_3 are same as defined in *case 1* and $E_4 = \{w_i\, w_{i+1}, uw_1, vw_{\ell-1} / 1\le i \le \ell-2\}$.

Consider the same vertex valued function as mentioned in case 1 added with
$f(w_i) = f(v_{\ell-1,\lfloor m/2 \rfloor}) + i$

It induces $f^*: E \to N$ defined by $f^*(uv) = |[f(\underline{u})]^2-[f(v)]^2|$, $\forall uv \in$ E, when ℓ is odd, we receive the edge labeling as same as in Case 1.

When ℓ is even, we get
for $1 \le j \le \lfloor m/2 \rfloor$
$f^*(uu_{1,j}) \equiv 0 \ (mod\ 4)$
$f^*(vv_{\ell-1,j}) \equiv 4 (mod\ 8)$
$f^*(uv_{1,j}) \equiv 3 \ (mod\ 4)$
$f^*(vu_{\ell-1,j}) \equiv 1 \ (mod\ 4)$

In addition to this, we attain the induced function f^* for E_4 as odd integer with strictly increasing sequence, if the vertex labeling of one end of the edge labeled with odd integer and the other end with even integer. Thus, in both the cases the entire ℓm edge labels are all distinct. Hence the $\theta[\ell^{[m]}]$ is SD graph. For example, the graph $\theta[6^{[6]}]$ and $\theta[6^{[7]}]$ are given in *Figure*. 5(a) & 5(b) respectively.

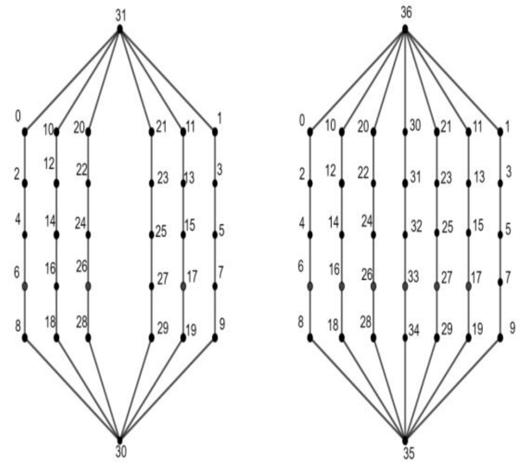

Fig 5:- SDL for $\theta[6^{[6]}]$ and $\theta[6^{[7]}]$

E. Theorem 3.5

The graph $H[\theta[\ell^{[m]}]]$ is a square difference graph.

➢ *Proof*

Let $G = H[\theta[\ell^{[m]}]]$ be the hanging generalized theta graph with $|V(G)| = m(\ell-1)+3$ and $|E(G)| = \ell m+1$.
Let us discuss the following two cases.

- *Case 1:*

m is even

Let $V(G) = V_1 \cup V_2 \cup V_3$, where
$V_1 = \{u_{i,j}, v_{i,j} \,/\, 1 \le i \le \ell-1, 1 \le j \le m/2\}$
$V_2 = \{u, v\}$
$V_3 = \{x\}$
and $E = \bigcup_{k=1}^{5} E_k$, where
For $1 \le i \le \ell-2$ & $1 \le j \le m/2$
$E_1 = \{u_{i,j}\, u_{i+1,j}\}$
$E_2 = \{v_{i,j}\, v_{i+1,j}\}$
$E_3 = \{uu_{1,j}\, uv_{1,j}\}$
$E_4 = \{vu_{\ell-1,j}, vv_{\ell-1,j}\}$
$E_5 = \{xu\}$
and the bijective function f on V is defined as
For $1 \le j \le m/2$, $1 \le i \le \ell-1$
$f(x) = 0$
$f(u) = m(\ell-1)+2$
$f(v) = m(\ell-1)+1$
$f(u_{i,j}) = 2(i-1)+2(\ell-1)(j-1)$ and
$f(v_{i,j}) = 2i-1+2(\ell-1)(j-1)$,

- *Case 2:*

when m is odd

Define $V(G) = = \bigcup_{k=1}^{4} V_k$, where V_i (i = 1, 2, 3) are same as in the *case 1* and $V_4 = \{w_i\}$.
Similarly $E(G) = \bigcup_{k=1}^{6} E_k$ where E_i (i = 1, 2,…, 5) are same as *case 1* and $E_6 = \{w_i\, w_{i+1} \,/\, 1 \le i \le \ell-2\}$. Let the vertex labeled function are same as mentioned in case 1 added to
$f(w_i) = f(u_{\ell-1,\lfloor m/2 \rfloor})+i$

For the above vertex labeling we receive the edge labels for $f^*(u_{i,j}\, u_{i+1,j}) \equiv 0 \pmod 8$ & $f^*(v_{i,j}\, v_{i+1,j}) \equiv 4 \pmod 8$ and also the edges E_3, E_4 are receiving the edge labels in the manner, 3(mod 4), 0(mod 4), 0(mod 8), 1(mod 4) then the apex $f^*(xu) = [m(\ell-1)+2]^2$ and for E_6 we get the same result as in case 2 of *theorem 3.4* . Thus the edge labeling are all distinct. Hence the $H[\theta[\ell^{[m]}]$ admit square difference labeling.

F. Theorem 3.6

Duplication of all the edges of C_n ($n \ge 3$) graph admits SDL.

➢ *Proof*

Consider the graph G obtained by duplicating of all the edges of C_n ($n \ge 3$) with
$V = \{x_i, x'_i \,/\, 1 \le i \le n\}$ and
$E = \{x_i x_{i+1}\} \cup \{x_i x'_i\} \cup \{x'_i x_{i+1}\} \cup \{x_n x_1\} \cup \{x'_n x_1\}$.

Let the vertex valued function g: $V \to \{0, 1, 2, \ldots 2n-1\}$ be defined as follows.
For $1 \le i \le n$,
$g(x_i) = 2i$
$g(x'_i) = 2i-1$

For the above labeling pattern, the induced function g^*: $E(G) \to N$ satisfies the condition of SD Labeling. Thus, the edges of G receives labels as,
$g^*(x_i x_{i+1}) = 8i-4$, $1 \le i \le n-1$
$g^*(x_n x_1) = (2n-2)^2$
$g^*(x_i x'_i) = 4i-3$ and $g^*(x'_i x_{i+1}) = 4i-1$, $1 \le i \le n$
$g^*(x'_n x_1) = (2n-1)^2$.

Thus all the edge labeling are distinct. i.e., $g^*(e_i) \neq g^*(e_j)$, $\forall e_i \neq e_j \in E(G)$. Hence the duplication of all the edges of C_n graph admits SD Labeling. The example is illustrated in *figure* 6.

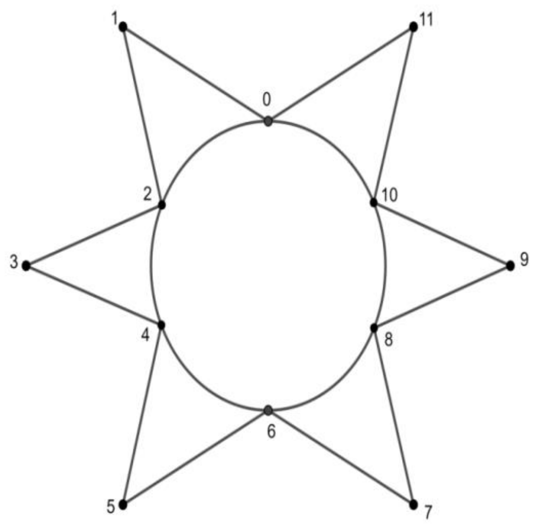

Fig 6:- SDL for Duplication of all the Edges of C_6

G. Theorem 3.7

The Square graph of comb admits square difference labeling.

➢ *Proof*

Let G be the square graph of comb and denoted by $(P_n \odot K_1)^2$with the vertex

$V = \{x_i , y_i \,/\, 1 \leq i \leq n\}$ and the edge $E = E_1 \cup E_2 \cup E_3$, where

$E_1 = \{x_i\, x_{i+1}, x_i\, y_{i+1}\}, i = 1,2,\ldots n\text{-}1$

$E_2 = \{x_ix_{i+2}, y_ix_{i+2}\}, i = 1$ to $n\text{-}2$

$E_3 = \{x_iy_i\}$.

Clearly the number of vertices and edges are $2n$ & $5n–1$ resp.,

Now, the vertex valued function f: V → {0, 1, 2, … 2n–1} be defined as,

for $1 \leq i \leq n$

$f(x_i) = i–1$

$f(y_i) = f(x_n) + i$

Then the induced edge function f^* yields the edge labeling as follows:

$f^*(x_ix_{i+1}) = 2i–1$&

$f^*(x_iy_{i+1}) = (n+1)^2 + 2(n+1)(i–1)$, for $1\leq i\leq n–1$

$f^*(x_ix_{i+2}) = 4i$

$f^*(y_ix_{i+2}) = (n^2–1) + (2n–2)(i–1)$, for $1 \leq i \leq n–2$

$f^*(x_iy_i) = n^2 + 2n(i–1)$, for $1 \leq i \leq n$.

Hence $f^*(e_i) \neq f^*(e_j)\ \forall e_i, e_j\in E(G)$. i.e., all the edge labeling are distinct.

Therefore, the square graph of comb $(P_n \odot K_1)^2$ admits SD Labeling. The example of $(P_n \odot K_1)^2$ is shown below.

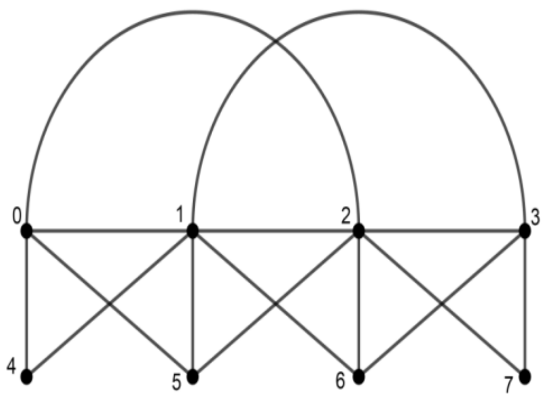

Fig 7:- $(P_4 \odot K_1)^2$ is SD Graph

H. Theorem 3.8

The graph $B^2_{n,n}$ is a square difference graph (n ≤ 4).

➢ *Proof*

Consider the graph $G = B^2_{n,n}$ with the vertex set $V = \{u, v, u_iv_i \,/\, 1\leq i\leq n\}$ and the edge set be

$E = \{uv, uu_i, vv_i, u_iv, v_iu\}$. Then $|V(G)| = 2n+2$& $|E(G)| = 4n+1$.

Now define a vertex value function $f : V\rightarrow \{0, 1, 2, \ldots 2n+1\}$as,

$f(u) = 0$

$f(v) = 1$

$f(u_i) = 2i$, &

$f(v_i) = 2i+1$, for $1\leq i\leq n$.

and the induced edge function f^* for the above labeling pattern, we get

$f^*(uu_{i+1}) = (2i)^2$

$f^*(uv) = 1$

$f^*(vu_{i+1}) = (2i)^2–1$

$f^*(vv_{i+1}) = (2i+1)^2 + 1$

$f^*(uv_{i+1}) = (2i+1)^2$

Thus the entire 4n+1 edges receive distinct labels which form an increasing sequence of positive integers. Hence the square graph of $B_{n,n}$ is square difference graph. SDL of $B^2_{5,5}$ is given below.

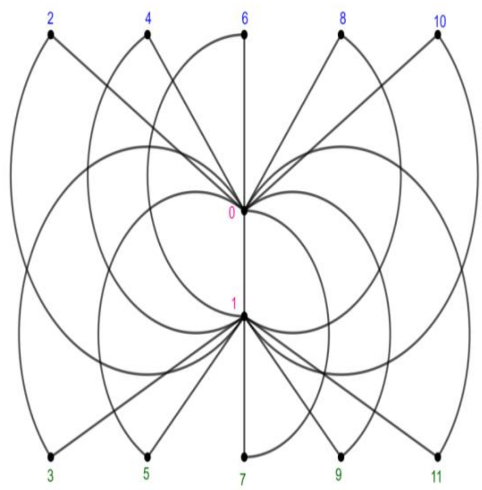

Fig 8:- $B^2_{5,5}$ is SDL

IV. CONCLUSION

In this paper, we investigated that the subdivision graph, $(K^{(1)}_{1,n}, K^{(2)}_{1,n})$, $(C_m \odot \overline{K}_n)$, $\theta[\ell^{[m]}]$, $H[\theta[\ell^{[m]}]]$, duplication of all the edges of C_n, $(P_n \odot K_1)^2$, $B^2_{n,n}$ are square difference graph.

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